

MONODROMY INVARIANTS AND POLARIZATION TYPES OF GENERALIZED KUMMER FIBRATIONS

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ABSTRACT. In this paper a monodromy invariant for isotropic classes on generalized Kummer type manifolds is constructed. This invariant is used to determine the polarization type of Lagrangian fibrations on such manifolds - a notion which was introduced in an earlier paper of the author. The result shows that the polarization type of a Lagrangian fibration of generalized Kummer type depends on the connected component of the moduli space.

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1. Introduction

The paper is a sequel of the the author’s paper [Wie16]. There the notion of a *polarization type of a Lagrangian fibration* was introduced.

Let $f : X \rightarrow B$ be a Lagrangian fibration. It is well known that all smooth fibers are abelian varieties even if X is not projective. Given a smooth fiber F an immediate question is to ask for natural polarizations on it which is by definition the first Chern class $H = c_1(L)$ of an ample line bundle L of F .

It is known that for each smooth fiber F one can find a Kähler class ω on X such that the restriction $\omega|_F$ is integral and primitive and hence defines a polarization on F . An ad-hoc definition of the *polarization type* of a Lagrangian fibration would be to set $\underline{d}(f) := \underline{d}(\omega|_F)$ where the latter one is the polarization type of the polarization on F given by $\omega|_F$. It follows that this does not depend on the chosen F and ω and that the polarization type stays constant in families of Lagrangian fibrations. Further $\underline{d}(f) = (1, \dots, 1)$ for every Lagrangian fibration of $K3^{[n]}$ -type. For a summary see also section 3.

The main purpose of the paper is to determine the polarization type of Lagrangian fibrations of generalized Kummer type.

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Theorem 1.1 (Theorem 7.1, Proposition 6.27) *Let $f : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration of generalized Kummer type. If $d = \text{Div}(\lambda)$ denotes the divisibility¹ of $\lambda = c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1))$, then d^2 divides $n + 1$ and we have for the polarization type*

$$\underline{d}(f) = \left(1, \dots, 1, d, \frac{n+1}{d}\right).$$

Further, for a fixed dimension $\dim X = 2n$, the divisibilities of classes λ as above which can appear for the generalized Kummer type, are exactly the positive integers d such that d^2 divides $n + 1$.

The proofs of Theorem 1.1 involve moduli theory of Lagrangian fibrations of generalized Kummer type, as for instance exploited in [Mar14]. The moduli theory appears in form of what is called a *monodromy invariant*.

Let X be an irreducible holomorphic symplectic manifold and consider the monodromy group $\text{Mon}^2(X)$. A *faithful monodromy invariant*, see section 5 and [Mar13, Def. 5.16], is a $\text{Mon}^2(X)$ -invariant map $\vartheta : I(X) \rightarrow \Sigma$ where $I(X) \subset H^2(X, \mathbb{Z})$ is a $\text{Mon}^2(X)$ -invariant subset and Σ is an arbitrary set, such that the induced map $I(X)/\text{Mon}^2(X) \rightarrow \Sigma$ is injective.

The following is a generalized Kummer analogue of E. Markman's monodromy invariant for the $\text{K3}^{[n]}$ case, see [Mar14, 2.].

Let X be of generalized Kummer type. For a fixed positive integer d , let denote $I_d(X) \subset H^2(X, \mathbb{Z})$ the set of all primitive isotropic classes with divisibility d . For the case that d^2 divides $n + 1$, let $\Sigma_{n,d}$ denote the set of isometry classes of pairs (H, w) such that H is a lattice isometric to the lattice $L_{n,d}$ which is defined in (5.11) and $w \in H$ is a primitive class with $(w, w) = 2n + 2$.

Theorem 1.2 (Theorem 5.1) *Let X be a generalized Kummer type manifold of dimension $2n$ and d a positive integer such that d^2 divides $n + 1$. There is a surjective faithful monodromy invariant*

$$\vartheta : I_d(X) \longrightarrow \Sigma_{n,d}$$

of the manifold X .

Structure of the paper. In section 2 we give a review of the theory of hyperkähler manifolds. Section 3 is a summary of the author's paper [Wie16] about the definition of the polarization type of a Lagrangian fibration. In section 4 an orbit of primitive isometric embeddings from the generalized Kummer lattice into the Mukai lattice (for a two torus) is constructed which is a main ingredient for the construction of the monodromy invariant which is done in the next section 5. Section 6 has the purpose to recall the construction of Beauville–Mukai systems of generalized Kummer type and to determine their polarization types. An excursion to the theory of Jacobians is needed, see subsection 6.7. Finally, we compute the polarization type of a Lagrangian fibration of generalized Kummer type in section 7.

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¹Here we mean with the divisibility $k = \text{Div}(\lambda)$, the largest positive number k , such that $(\lambda, \cdot)/k$ is an integral form.

2. Hyperkähler Manifolds and their fibrations

In this section we recall the basic facts about irreducible holomorphic symplectic manifolds and their fibrations which are Lagrangian.

Definition 2.1 A compact Kähler manifold X is called *hyperkähler* or *irreducible holomorphic symplectic* if X is simply connected and $H^0(X, \Omega_X^2)$ is generated by a nowhere degenerate holomorphic two-form σ .

Note that σ is automatically symplectic since every holomorphic form on a compact Kähler manifold is closed.

The most basic example is provided by the Douady space $S^{[n]}$ of n points for a K3 surface S which parametrizes zero-dimensional subspaces of S of length n . A. Beauville [Bea84] showed that $S^{[n]}$ is an irreducible holomorphic symplectic manifold of dimension $2n$.

In this paper we are more interested in the following example. We start with a complex two-torus S and take the Douady space $S^{[n+1]}$. This is holomorphic symplectic, but not simply connected. Then one uses the *Douady–Barlet map*

$$\rho : S^{[k]} \longrightarrow S^{(k)}, \quad Z \longmapsto \sum_{z \in Z} (\dim_{\mathbb{C}} \mathcal{O}_{Z,z}) z$$

which is a resolution of singularities of the *symmetric product* $S^{(k)} := (S \times \cdots \times S) / \Sigma_k$ to obtain a morphism

$$S^{[n+1]} \xrightarrow{\rho} S^{(n+1)} \xrightarrow{+} S$$

where the last map is summation in S . By A. Beauville [Bea84] the fiber $S^{[[n]]} = K_n(S)$ over 0 is an irreducible holomorphic symplectic manifold of dimension $2n$, called *generalized Kummer manifold* as $S^{[[1]]}$ is the usual Kummer K3 surface.

An irreducible holomorphic symplectic manifold is of *K3^[n]-type* or of *generalized Kummer type* if it is deformation equivalent to $S^{[n]}$ for a K3 surface S or to $S^{[[n]]}$ for a two-torus S , respectively.

The second cohomology $H^2(X, \mathbb{Z})$ of any irreducible holomorphic symplectic manifold X admits the well known *Beauville–Bogomolov–Fujiki* quadratic form q_X which is non-degenerate and of signature $(3, b_2(X) - 3)$, see [GHJ03, 23.3]. The associated bilinear form is denoted by (\cdot, \cdot) . On an abstract lattice we also denote the bilinear form by (\cdot, \cdot) . The lattice $H^2(X, \mathbb{Z})$ with the Beauville–Bogomolov–Fujiki form is a deformation invariant of the manifold X . For manifolds of generalized Kummer type this lattice is isometric to the abstract *generalized Kummer lattice*

$$\Lambda = U^{\oplus 3} \oplus \langle -(2 + 2n) \rangle,$$

see [Bea84, Prop. 8] where $\langle -(2 + 2n) \rangle$ denotes the lattice of rank one with generator l such that $(l, l) = 2 + 2n$ and U the unimodular rank two hyperbolic lattice.

A *marking* on an irreducible holomorphic manifold X is the choice of an isometry $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$. The tuple (X, η) is then called a *marked pair* or a *marked irreducible holomorphic symplectic manifold*.

If X is a fixed irreducible holomorphic symplectic manifold set $\Lambda := H^2(X, \mathbb{Z})$ and consider the Kuranishi family $\pi : \mathfrak{X} \rightarrow \text{Def}(X)$ with $\mathfrak{X}_0 := \pi^{-1}(0) = X$. We will view the base $\text{Def}(X)$ sometimes as a germ but also as a representative which we choose small enough i.e. simply connected. Then by Ehresmann’s theorem we can choose a

trivialization $\Sigma : R^2\pi_*\mathbb{Z} \rightarrow \Lambda_{\text{Def}(X)}$ also called a *marking* and define the *local period map* by

$$\mathcal{P} : \text{Def}(X) \longrightarrow \mathbb{P}(\Lambda_{\mathbb{C}}), \quad t \longmapsto [\Sigma_t(H^{2,0}(\mathfrak{X}_t))]$$

where $\Lambda_{\mathbb{C}} := \Lambda \otimes \mathbb{C}$. It takes values in the *period domain of type Λ* [GHJ03, 22.3] namely

$$\Omega_{\Lambda} := \{p \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid (p, p) = 0 \text{ and } (p, \bar{p}) > 0\}$$

which is connected since the signature of q_X is $(3, \text{rk } \Lambda - 3)$.

Theorem 2.1 (LOCAL TORELLI, [Bea84], 8.) *The period map $\mathcal{P} : \text{Def}(X) \rightarrow \Omega_{\Lambda}$ is an open embedding.*

Two marked pairs (X_i, η_i) , $i = 1, 2$, are called *isomorphic* if there is an isomorphism $f : X_1 \rightarrow X_2$ such that $\eta_2 = \eta_1 \circ f^*$. There exists a *moduli space of marked pairs* $\mathfrak{M}_{\Lambda} := \{(X, \eta) \text{ marked pair}\} / \cong$ which can be constructed by glueing all deformation spaces $\text{Def}(X)$ of irreducible holomorphic symplectic manifolds X with $H^2(X, \mathbb{Z})$ isometric to Λ . This gives a non-Hausdorff complex manifold of dimension $\text{rk } \Lambda - 2$. The *global period mapping*

$$\mathcal{P} : \mathfrak{M}_{\Lambda} \longrightarrow \Omega_{\Lambda}, \quad (X, \eta) \longmapsto [\eta(H^{2,0}(X))]$$

is locally given by $\mathcal{P} : \text{Def}(X) \rightarrow \Omega_{\Lambda}$ and hence is again a local biholomorphism by the Local Torelli. If one takes an arbitrary connected component $\mathfrak{M}_{\Lambda}^{\circ}$ of \mathfrak{M}_{Λ} then by a result of D. Huybrechts [GHJ03, Prop. 25.12] the restriction $\mathcal{P} : \mathfrak{M}_{\Lambda}^{\circ} \rightarrow \Omega_{\Lambda}$ is surjective.

If L denotes a line bundle on X by abuse of notation we also denote the universal family of the pair (X, L) by $\pi : \mathfrak{X}_L \rightarrow \text{Def}(X, L)$ which comes with an universal line bundle \mathcal{L} on \mathfrak{X}_L such that $(\mathfrak{X}_L)_0 = X$ and $\mathcal{L}_0 = L$, see [Bea84, Cor. 1]. We consider again $\text{Def}(X, L)$ as a germ but as well as a proper space. A representative of $\text{Def}(X, L)$ is locally given by $(c_1(L), \cdot) = 0$ in Ω_{Λ} hence it is a smooth hypersurface in $\text{Def}(X)$, see [GHJ03, 26.1] and one defines \mathfrak{X}_L as the preimage of it under π . The family $\pi : \mathfrak{X}_L \rightarrow \text{Def}(X, L)$ is the restriction of the Kuranishi family $\pi : \mathfrak{X} \rightarrow \text{Def}(X)$ to \mathfrak{X}_L and $\text{Def}(X, L)$.

2.2. Lagrangian fibrations. Due to D. Matsushita much is known about non-trivial fiber structures on irreducible holomorphic symplectic manifolds.

Theorem 2.2 (MATSUSHITA, [Mat99], [Mat00], [Mat01], [Mat03]) *Let $f : X \rightarrow B$ be a surjective holomorphic map with connected fibers from an irreducible holomorphic symplectic manifold X of dimension $2n$ to a normal complex space B such that $0 < \dim B < 2n$. Then the following statements hold.*

- (i) *B is projective of dimension n and its Picard number is $\rho(B) = 1$.*
- (ii) *For all $t \in B$ the fiber $X_t := f^{-1}(t)$ is Lagrangian subspace i.e. $\sigma|_{X_t^{\text{reg}}} = 0$ where X_t^{reg} denotes the smooth part of X_t .*
- (iii) *If X_t is smooth then it is a projective complex torus i.e. an abelian variety.*

Such a fibration $f : X \rightarrow B$ as in the Theorem is called a *Lagrangian fibration*. If X is a $\text{K3}^{[n]}$ -type manifold then we call $f : X \rightarrow B$ a *$\text{K3}^{[n]}$ -type fibration*.

If the base of the Lagrangian fibration is smooth even more is known due to a deep result of J.-M. Hwang which was recently slightly generalized by C. Lehn and D. Greb to the non-projective case.

Theorem 2.3 (HWANG, [Hwa08], [GL14]) *Let $f : X \rightarrow B$ be a Lagrangian fibration such that B is smooth and $\dim X = 2n$. Then $B \cong \mathbb{P}^n$.*

If $f : X \rightarrow B$ is a $K3^{[n]}$ -type fibration then E. Markman [Mar11, Thm. 1.3, Rem. 1.8] in combination with a result of D. Matsushita [Mat13, Thm. 1.2, Cor. 1.1] has shown that $B \cong \mathbb{P}^n$ without assuming smoothness of B . By [Yos12, Appendix] also in combination with [Mat13, Thm. 1.2, Cor. 1.1] this holds for Lagrangian fibrations of generalized Kummer type.

The basic example of a Lagrangian fibration on a generalized Kummer manifold can be obtained as follows. Let $f : S \rightarrow E$ be a surjective holomorphic map where S is a two torus and E is an elliptic curve. With use of the Douady–Barlet map we have a map

$$S^{[[n]]} \hookrightarrow S^{[n+1]} \xrightarrow{\rho} S^{(n+1)} \xrightarrow{f \times \cdots \times f} E^{(n+1)} \cong \mathbb{P}^n \times E.$$

This map and the projection from $\mathbb{P}^n \times E$ to \mathbb{P}^n defines a Lagrangian fibration $S^{[[n]]} \rightarrow \mathbb{P}^n$ by Matsushita’s Theorem 2.2. Let F denote a smooth fiber of p , then the fiber of the Lagrangian fibration $S^{[[n]]} \rightarrow \mathbb{P}^n$ is isomorphic to the abelian subvariety of F^{n+1} given by the equation $x_1 + \cdots + x_{n+1} = 0$ for $(x_1, \dots, x_{n+1}) \in F^{n+1}$.

Note that two-dimensional Lagrangian fibrations are exactly the elliptic K3 surfaces.

Definition 2.3 (i) A family of Lagrangian fibrations over a connected complex space S with finitely many irreducible components is an S -morphism

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\phi} & P \\ & \searrow & \swarrow \\ & S & \end{array}$$

where $\mathcal{X} \rightarrow S$ is a family of irreducible holomorphic symplectic manifolds and $P \rightarrow S$ is a family of projective varieties such that for every $s \in S$ the restriction $\phi|_{\mathcal{X}_s} : \mathcal{X}_s \rightarrow P_s$ to the irreducible holomorphic symplectic manifold \mathcal{X}_s is a Lagrangian fibration.

(ii) Two Lagrangian fibrations f_1 and f_2 are *deformation equivalent* if there is a family of Lagrangian fibrations ϕ over a connected complex space S containing f_1 and f_2 i.e. there are points $t_i \in S$ such that $\phi_{t_i} = f_i$, $i = 1, 2$.

Definition 2.4 [Mar13, 5.2] Let X_i , $i = 1, 2$, denote two irreducible holomorphic symplectic manifolds, L_i holomorphic line bundles on X_i and e_i classes in $H^2(X_i, \mathbb{Z})$.

(i) The pairs (X_1, e_1) and (X_2, e_2) are called *deformation equivalent* if there exists a family $\pi : \mathcal{X} \rightarrow S$ of irreducible holomorphic symplectic manifolds over a connected complex space S with finitely many irreducible components, a section e of $R^2\pi_*\mathbb{Z}$, points t_i in S such that $\mathcal{X}_{t_i} = X_i$ and $e_{t_i} = e_i$.

(ii) The pairs (X_1, L_1) and (X_2, L_2) are called *deformation equivalent* if there exists a family $\pi : \mathcal{X} \rightarrow S$ of irreducible holomorphic symplectic manifolds over a connected complex space S with finitely many irreducible components, a line bundle \mathcal{L} on \mathcal{X} , points t_i in S such that $\mathcal{X}_{t_i} = X_i$ and $\mathcal{L}_{\mathcal{X}_{t_i}} = L_i$.

Remark 2.5 Note that we can reformulate (ii) of Definition 2.4 as the following.

- The pairs (X_1, L_1) and (X_2, L_2) are called *deformation equivalent* if there exists a family $\pi : \mathcal{X} \rightarrow S$ of irreducible holomorphic symplectic manifolds over a connected complex space S with finitely many irreducible components, a section e of $R^2\pi_*\mathbb{Z}$ which is everywhere of Hodge type $(1, 1)$, points t_i in S such that $\mathcal{X}_{t_i} = X_i$ and $e_{t_i} = c_1(L_i)$.

Clearly, $e_t := c_1(\mathcal{L}_t)$ would give such a section. Conversely, given a section e as in the alternative definition, we get a line bundle L_t on \mathcal{X}_t corresponding to $e_t \in H^{1,1}(\mathcal{X}_t, \mathbb{Z})$ with respect to the isomorphism $\text{Pic}(\mathcal{X}_t) \cong H^{1,1}(\mathcal{X}_t, \mathbb{Z})$ since \mathcal{X}_t is irreducible holomorphic symplectic. Then the Kuranishi family of the pair (\mathcal{X}_t, L_t) gives an universal line bundle on the respective total space for every $t \in S$. Those line bundles glue to a line bundle \mathcal{L} on \mathcal{X} with the property $c_1(\mathcal{L}_t) = e_t$.

Proposition 2.6 *Let $f_i : X_i \rightarrow \mathbb{P}^n$, $i = 1, 2$, denote two Lagrangian fibrations of generalized Kummer or $K3^{[n]}$ -type and set $L_i := f_i^* \mathcal{O}_{\mathbb{P}^n}(1)$. The Lagrangian fibrations f_i are deformation equivalent in sense of Definition 2.3, if and only if the pairs (X_i, L_i) are deformation equivalent.*

Proof: The proof is exactly the same as in [Wie16, Prop. 3.9]. There everything is stated for the $K3^{[n]}$ -type, but it carries over to the generalized Kummer type word by word. \square

Lemma 2.7 [Wie16, Lem. 3.5] *Let $f : X \rightarrow B$ be a Lagrangian fibration and let $L := f^* A$ be the pullback of a line bundle A on B .*

- (i) *L is isotropic with respect to the Beauville–Bogomolov quadratic form.*
- (ii) *If A admits nontrivial sections then L is nef.*
- (iii) *If X is of $K3^{[n]}$ or generalized Kummer type, then L is primitive.*

Proof: The first two statements are contained in [Wie16, Lem. 3.5]. The third statement is formulated for the $K3^{[n]}$ -type, but the proof works also in the generalized Kummer case with use of [Mat13, Cor. 1.1]. \square

2.8. Orientation. We summarize section 4. of [Mar11].

Let $b_2 > 0$ a positive integer and Λ be an even lattice of signature $(3, b_2 - 3)$. Define

$$\tilde{\mathcal{C}}_\Lambda := \{x \in \Lambda_{\mathbb{R}} \mid (x, x) > 0\}.$$

We have the following.

Lemma 2.9 [Mar11, Lem. 4.1] *If $W \subset \Lambda_{\mathbb{R}}$ is a three dimensional subspace such that the bilinear form of Λ is positive definite on it, then $W \setminus \{0\}$ is a deformation retract of $\tilde{\mathcal{C}}_\Lambda$. Therefore $H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ is a free abelian group of rank one. The reflection R_u for $u \in \Lambda$ with $(u, u) \neq 0$ given by*

$$R_u(x) := (x, x) - 2 \frac{(u, x)}{(u, u)} u,$$

acts on $H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z})$

- *as $+1$ if $(e, e) < 0$ and*
- *as -1 if $(e, e) > 0$,*

therefore it defines a generator of $H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z})$.

In particular, the Lemma implies that $\tilde{\mathcal{C}}_\Lambda$ is connected, as $H_0(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z}) = H_0(W \setminus \{0\}, \mathbb{Z}) = \mathbb{Z}$.

Definition 2.10 *An orientation of $\tilde{\mathcal{C}}_\Lambda$ is a choice of a generator of $H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$.*

By speaking of oriented isometries of the lattice Λ , we mean isometries which preserve the orientation of $\tilde{\mathcal{C}}_\Lambda$ in sense of the definition above: every isometry $g :$

$\Lambda \rightarrow \Lambda$ induces a homeomorphism $g : \tilde{\mathcal{C}}_\Lambda \rightarrow \tilde{\mathcal{C}}_\Lambda$, therefore we have a morphism

$$(2.11) \quad \begin{aligned} \mathrm{O}(\Lambda) &\longrightarrow \mathrm{Aut}(H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z})) \cong \{\pm 1\} \\ g &\longmapsto g^* . \end{aligned}$$

Definition 2.12 The morphism in (2.11) above is also called *spinor norm*. Its kernel is denoted by $\mathrm{O}^+(\Lambda)$ and isometries in it are called *orientation preserving*.

For a period $p \in \Omega_\Lambda$ let $\Lambda(p)$ denote the integral Hodge structure of weight two of Λ determined by the period p , that is

$$(2.13) \quad \Lambda^{2,0}(p) = p, \quad \Lambda^{0,2}(p) = \bar{p} \quad \text{and} \quad \Lambda^{1,1}(p) = \{x \in \Lambda_\mathbb{C} \mid (x, p) = (x, \bar{p}) = 0\} .$$

As in the geometric situation, we also set

$$\Lambda^{1,1}(p, R) := \{x \in \Lambda_R \mid (x, p) = 0\}$$

for $R \in \{\mathbb{Z}, \mathbb{R}\}$. Further consider the set

$$(2.14) \quad \mathcal{C}'_p := \left\{x \in \Lambda^{1,1}(p, \mathbb{R}) \mid (x, x) > 0\right\} .$$

The restriction of the bilinear form to $\Lambda^{1,1}(p, \mathbb{Z})$ has signature $(1, b_2 - 3)$. Therefore \mathcal{C}'_p has two connected components.

Let x be in \mathcal{C}'_p with $p = \mathbb{C} \cdot \sigma$. We can define a subspace

$$(2.15) \quad W_x := \mathrm{Re}(p) \oplus \mathrm{Im}(p) \oplus \mathbb{R} \cdot x$$

of $\Lambda_\mathbb{R}$ such that the bilinear form is positive definite on it. The subspace W_x of $\Lambda_\mathbb{R}$ defines a generator of $H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ in the following way.

The subvector space W_x has the canonical ordered basis

$$(2.16) \quad (\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), x) ,$$

which defines an orientation in the ordinary sense i.e. a volume form $\beta(\sigma) := \mathrm{Re}(\sigma)^* \wedge \mathrm{Im}(\sigma)^* \wedge x^*$ of the manifold $W_x \setminus \{0\}$. The orientation $\beta(\sigma)$ does not depend on the choice of σ , indeed we have $\beta(\lambda\sigma) = |\lambda|\beta(\sigma)$ for any $\lambda \in \mathbb{C}$. Take the two sphere $\mathbb{S}^2 \subset W_x \setminus \{0\}$ in W_x . It is well known, that the basis (2.16) gives a volume form on \mathbb{S}^2 by restricting the two form

$$x_1 \mathrm{im}(\sigma)^* \wedge x^* + x_2 x^* \wedge \mathrm{Re}(\sigma)^* + x_3 \mathrm{Re}(\sigma)^* \wedge \mathrm{im}(\sigma)^*$$

to \mathbb{S}^2 , where x_1, x_2, x_3 are the standard coordinates with respect to the basis (2.16). Use

$$(2.17) \quad H^2(\mathbb{S}^2, \mathbb{Z}) = H^2(W_x \setminus \{0\}, \mathbb{Z}) = H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z})$$

to obtain a generator of $H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z})$ i.e. an orientation in sense of Definition 2.10. Obviously we end up with the other generator, if we change the orientation of W_x given by the basis (2.16).

Principle 2.18 An element x in \mathcal{C}'_p for a period $p \in \Omega_\Lambda$ determines a generator of $H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ i.e. an orientation of $\tilde{\mathcal{C}}_\Lambda$. The two generators are distinguished by the two connected components of \mathcal{C}'_p . Therefore a connected component of \mathcal{C}'_p determines an orientation of $\tilde{\mathcal{C}}_\Lambda$.

2.19. The geometric situation. Let \mathfrak{M}_Λ denote the moduli space of isomorphism classes of marked pairs (X, η) of type Λ i.e. X is an irreducible holomorphic

symplectic manifold and $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ is a marking. Choose a connected component $\mathfrak{M}_\Lambda^\circ$ of \mathfrak{M}_Λ . Recall that for $(X, \eta) \in \mathfrak{M}_\Lambda^\circ$ there is a canonical choice for the connected component of

$$\mathcal{C}'_X := \{x \in H^{1,1}(X, \mathbb{R}) \mid (x, x) > 0\}$$

namely the *positive cone* \mathcal{C}_X which contains the Kähler cone \mathcal{K}_X of X . Therefore, by Principle 2.18

$$\tilde{\mathcal{C}}_X := \tilde{\mathcal{C}}_{H^2(X, \mathbb{Z})} = \{x \in H^2(X, \mathbb{R}) \mid (x, x) > 0\}$$

has a *natural orientation*, which determines an orientation in sense of Definition 2.10 of $\tilde{\mathcal{C}}_\Lambda$ via the homeomorphism $\eta : \tilde{\mathcal{C}}_X \cong \tilde{\mathcal{C}}_\Lambda$.

Definition 2.20 We will refer to the orientation of $\tilde{\mathcal{C}}_\Lambda$ (in sense of Definition 2.10) which is induced by the marking η and the natural orientation of $\tilde{\mathcal{C}}_X$ for some (hence for all) marked pair (X, η) in $\mathfrak{M}_\Lambda^\circ$, as the orientation *compatible* to the connected component $\mathfrak{M}_\Lambda^\circ$ of the moduli of marked pairs.

Consider the period map

$$\mathcal{P} : \mathfrak{M}_\Lambda^\circ \longrightarrow \Omega_\Lambda, \quad (X, \eta) \longmapsto [\eta(H^{2,0}(X))]$$

and set $p := \mathcal{P}(X, \eta)$. Then $\eta(H^{1,1}(X, \mathbb{R})) = \Lambda^{1,1}(p, \mathbb{R})$. An orientation of $\tilde{\mathcal{C}}_\Lambda$ determines a connected component

$$(2.21) \quad \mathcal{C}_p \subset \mathcal{C}'_p$$

of \mathcal{C}'_p by Principle 2.18. Equivalently, we can characterize the orientation compatible to $\mathfrak{M}_\Lambda^\circ$ by the condition $\eta(\mathcal{C}_X) = \mathcal{C}_p$ for all $(X, \eta) \in \mathfrak{M}_\Lambda^\circ$ with $p = \mathcal{P}(X, \eta)$.

2.22. $\Omega_{\lambda^\perp}^+$ for an isotropic class. For the following see also [Mar14, 4.3]. We are still in the setting of 2.19. Let $\lambda \in \Lambda$ be a nontrivial isotropic class. We define a hyperplane section

$$(2.23) \quad \Omega_{\lambda^\perp} := \Omega_\Lambda \cap \lambda^\perp = \{p \in \Omega_\Lambda \mid (p, \lambda) = 0\}.$$

Note that the bilinear form on $\lambda^\perp \subset \Lambda_\mathbb{R}$ is degenerate since λ is isotropic. The hyperplane section Ω_{λ^\perp} has two connected components and we can still obtain a natural connected component of it from the geometrical situation in the following way.

For $p \in \Omega_{\lambda^\perp}$, λ belongs to $\Lambda^{1,1}(p, \mathbb{R})$ and is contained in the boundary of one of the connected components of \mathcal{C}'_p since λ is isotropic. For $(X, \eta) \in \mathfrak{M}_\Lambda^\circ$, either $\eta^{-1}(\lambda)$ or $\eta^{-1}(-\lambda)$ belongs to $\partial\mathcal{C}_X$. We assume that the former is the case, otherwise take $-\lambda$. Then consider only periods p in Ω_{λ^\perp} such that λ belongs to the closure of the distinguished connected component \mathcal{C}_p in $\Lambda^{1,1}(p, \mathbb{R})$, see (2.21), determined by the orientation of $\tilde{\mathcal{C}}_\Lambda$ compatible to $\mathfrak{M}_\Lambda^\circ$ i.e.

$$(2.24) \quad \Omega_{\lambda^\perp}^+ := \{p \in \Omega_{\lambda^\perp} \mid \lambda \in \partial\mathcal{C}_p\}$$

which is one of the connected components of Ω_{λ^\perp} . Note that the only common element of the closures of the connected components of Ω_{λ^\perp} is the null vector, therefore $\Omega_{\lambda^\perp}^+$ of (2.24) is indeed one of the connected components of Ω_{λ^\perp} . We refer to $\Omega_{\lambda^\perp}^+$ as the *compatible* connected component of Ω_{λ^\perp} with respect to the chosen connected component $\mathfrak{M}_\Lambda^\circ$ of the moduli of marked pairs.

2.25. Monodromy. We recall some basic definitions and state G. Mongardi's monodromy result [Mon14].

Definition 2.26 Let X_i , $i = 1, 2$, be two irreducible holomorphic symplectic manifolds. An isometry $P : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ is called a *parallel transport operator* if there exists a family $\pi : \mathcal{X} \rightarrow S$ of irreducible holomorphic symplectic manifolds, points t_i such that $\mathcal{X}_{t_i} = X_i$ and a continuous path γ such that the parallel transport P_γ along γ in the local system $R^2\pi_*\mathbb{Z}$ coincides with P . For $X := X_1 = X_2$ it is also called a *monodromy operator* and the subgroup $\text{Mon}^2(X)$ of $O(H^2(X, \mathbb{Z}))$ generated by monodromy operators is called the *monodromy group*.

Let Λ denote a non-degenerate lattice of signature $(3, b_2 - 3)$.

Definition 2.27 Let $\mathcal{W}(\Lambda)$ denote the subgroup of $O^+(\Lambda)$ consisting of orientation preserving isometries acting as ± 1 on the discriminant Λ^\vee/Λ . Denote by

$$\chi : \mathcal{W}(\Lambda) \rightarrow \{\pm 1\}$$

the associated character. We also write $\mathcal{W}(X) := \mathcal{W}(H^2(X, \mathbb{Z}))$ for an irreducible holomorphic manifold X .

For a class $u \in \Lambda$ with $(u, u) \neq 0$ we have the rational reflection $R_u : \Lambda \rightarrow \Lambda$ defined by

$$(2.28) \quad R_u(x) := x - 2 \frac{(u, x)}{(u, u)} u.$$

If $(u, u) < 0$, then by Lemma 2.9 the reflection R_u is orientation preserving in sense of Definition 2.12 i.e. contained in $O^+(\Lambda_\mathbb{Q})$.

Definition 2.29 Let Λ be a non-degenerate lattice of signature $(3, b_2 - 3)$. For a class $u \in \Lambda$ with $(u, u) \neq 0$, denote $\rho_u : \Lambda_\mathbb{Q} \rightarrow \Lambda_\mathbb{Q} \in O^+(\Lambda_\mathbb{Q})$ the orientation preserving isometry defined by

$$\rho_u := \begin{cases} R_u & \text{if } (u, u) < 0, \\ -R_u & \text{if } (u, u) > 0. \end{cases}$$

Remark 2.30 (i) If $(u, u) = \pm 2$, then R_u and ρ_u define honest integral isometries $\Lambda \rightarrow \Lambda$.

(ii) The action of R_u on Λ^\vee for a $h \in \Lambda^\vee$ is

$$R_u(h)(x) = h(R_u(x)) = h(x) - (2 \frac{h(u)}{(u, u)} u, x),$$

i.e. $R_u(h) = h \pmod{\Lambda}$, hence for $(u, u) = \pm 2$ the isometry ρ_u is contained in $\mathcal{W}(\Lambda)$. More precisely we have

$$\chi(\rho_u) = \begin{cases} +1 & \text{if } (u, u) < 0, \\ -1 & \text{if } (u, u) > 0. \end{cases}$$

(iii) The isometry R_u satisfies $R_u(u) = -u$ and $R_u|_{u^\perp} = \text{id}_{u^\perp}$, hence we have for the determinant $\det(R_u) = -1$. Therefore

$$\det(\rho_u) = \begin{cases} -1 & \text{if } (u, u) < 0, \\ (-1)^{b_2+1} & \text{if } (u, u) > 0. \end{cases}$$

Note that for the K3^[n] and generalized Kummer case b_2 is odd and for the O'Grady examples b_2 is even.

Theorem 2.4 (MONGARDI, [Mon14, Thm. 2.3]) *Let X be a generalized Kummer n -type manifold. Then $\text{Mon}^2(X)$ consists precisely of orientation preserving isometries $g \in \mathcal{W}(X)$ such that $\chi(g) \cdot \det(g) = 1$.*

In particular, for a generalized Kummer manifold X , $\text{Mon}^2(X)$ is an index 2 subgroup of $\mathcal{W}(X)$ as $|\mathcal{W}(X)/\text{Mon}^2(X)| = |\text{im}(\det \cdot \chi)| = 2$.

Corollary 2.31 *For a generalized Kummer type manifold X , the monodromy group $\text{Mon}^2(X)$ is an index 2 subgroup of $\mathcal{W}(X)$. The orientation preserving isometry $\rho_u \in \mathcal{W}(X)$ for a class $u \in H^2(X, \mathbb{Z})$ with $(u, u) = \pm 2$ defined in Definition 2.29 is never contained in $\text{Mon}^2(X)$.*

Proof: The first statement we have just discussed. The second statement follows from Remark 2.30 (ii) and (iii). \square

3. Polarization types of Lagrangian fibrations

The author introduced the following notion in [Wie16]. Let $f : X \rightarrow B$ be a Lagrangian fibration. We know that all smooth fibers are abelian varieties by Theorem 2.2, even if X is not projective. For an abelian variety F of dimension $\dim F = n$, there is a well known classical notion of a *polarization*, cf. [BL03, p. 70], which is by definition the first Chern class $H = c_1(L)$ of an ample line bundle L of F . Often one calls the ample line bundle L a polarization. Furthermore, one can associate to such a polarization a *type*, which is a tuple

$$\underline{d}(L) = (d_1, \dots, d_n)$$

of positive integers such that d_i divides d_{i+1} , cf. [BL03, p. 70].

Given a smooth fiber F of the Lagrangian fibration f we want to consider polarization on it induced from X . First of all, it is not clear, how to obtain a polarization on a smooth fiber F of the Lagrangian fibration $f : X \rightarrow B$ if X is not projective. However, due to the following statement, which is related to an observation of C. Voisin [Cam06, Prop. 2.1], it is always possible.

Proposition 3.1 [Wie16, Prop 4.3] *For any smooth fiber F there is a Kähler class ω on X such that the restriction $\omega|_F$ is integral and primitive.*

Such a class ω is called *special Kähler class* (with respect to F) and defines a polarization $\omega|_F$ on the abelian variety F in the sense above. This polarization one can associate its type $\underline{d}(\omega|_F) := (d_1, \dots, d_n)$ where again d_i are positive integers such that d_i divides d_{i+1} .

Definition 3.2 The *polarization type* of a Lagrangian fibration $f : X \rightarrow B$ is

$$\underline{d}(f) := \underline{d}(\omega|_F) = (d_1, \dots, d_n).$$

This definition seems to be a bit ad-hoc, but it is convenient for a short introduction. The following statements were shown in [Wie16].

Theorem 3.1 [Wie16, Section 4] *Let $f : X \rightarrow B$ be a Lagrangian fibration with $\dim X = 2n$. Then the following statements hold.*

- (i) [Wie16, Prop. 4.7] *The polarization type $\underline{d}(f)$ is well defined i.e. does not depend on the chosen smooth fiber and the chosen special Kähler class (with respect to this fiber) and is a primitive vector in \mathbb{Z}^n .*
- (ii) [Wie16, Thm. 4.9] *The polarization type is a deformation invariant of the fibration i.e. if $f' : X' \rightarrow B'$ is a Lagrangian fibration deformation equivalent to f , then $\underline{d}(f) = \underline{d}(f')$.*

- (iii) [Wie16, Prop. 4.6, Prop. 4.10] Let B° denote the subset of B which parametrizes the smooth fibers. Then there exists a family of special Kähler classes, that is a map $\alpha : B^\circ \rightarrow \mathcal{H}$ where $\mathcal{H} \subset (R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_B)|_{B^\circ}$ is a subbundle and $\alpha(t)$ is a special Kähler class with respect to the smooth fiber X_t for every $t \in B^\circ$. In particular $\underline{d}(\alpha(t)) = \underline{d}(f)$ for every $t \in B^\circ$.
- (iv) [Wie16, Prop. 4.6] The family of special Kähler classes α induces a holomorphic map, called moduli map,

$$\begin{aligned} \phi : B^\circ &\longrightarrow \mathcal{A}_{\underline{d}(f)}, \\ t &\longmapsto (X_t, \alpha(t)) \end{aligned}$$

where $\mathcal{A}_{\underline{d}(f)}$ denotes the moduli space of $\underline{d}(f)$ polarized abelian varieties.

- (v) [Wie16, Thm. 6.1] Let $f : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration of $K3^{[n]}$ -type. Then $\underline{d}(f) = (1, \dots, 1)$.

In this paper, we want to determine the polarization type of a Lagrangian fibration of generalized Kummer type.

4. An orbit of primitive isometric embeddings

The main ingredient for the construction of a monodromy invariant for isotropic classes in the second cohomology of a generalized Kummer manifold is a monodromy invariant orbit of primitive isometric embeddings of the Kummer-type lattice into the Mukai lattice.

The group of isometries $O(\tilde{\Lambda})$ of the Mukai lattice and $O(\Lambda)$ acts on the set $O(\Lambda, \tilde{\Lambda})$ of primitive isometric embeddings $\iota : \Lambda \hookrightarrow \tilde{\Lambda}$ of the lattice Λ into $\tilde{\Lambda}$ by composition i.e. for $g \in O(\Lambda)$ and $\tilde{g} \in O(\tilde{\Lambda})$ one sets $g \cdot \iota := \iota \circ g$ and $\tilde{g} \cdot \iota := \tilde{g} \circ \iota$.

Definition 4.1 Let $\iota \in O(\Lambda, \tilde{\Lambda})$ be a primitive isometric embedding. An element $g \in O(\Lambda)$ leaves the $O(\tilde{\Lambda})$ -orbit $[\iota] = O(\tilde{\Lambda})\iota$ invariant if $g \cdot [\iota] := [\iota \circ g] = [\iota]$ i.e. if there exists $\tilde{g} \in O(\tilde{\Lambda})$ such that $\tilde{g} \circ \iota = \iota \circ g$. The orbit is called *monodromy invariant* if $\text{Mon}^2(X) \cdot [\iota] = [\iota]$ i.e. all elements in $\text{Mon}^2(X)$ leave the orbit $[\iota]$ invariant.

Remark 4.2 Let $\iota : \Lambda \hookrightarrow \tilde{\Lambda}$ denote a primitive isometric embedding. If X is a generalized Kummer type manifold then $\iota(\Lambda)^\perp = \langle v \rangle$ is of rank 1 since the Mukai lattice is of rank 8 and the Kummer type lattice is of rank 7. An isometry $\tilde{g} \in O(\tilde{\Lambda})$ with $\iota \circ g = \tilde{g} \circ \iota$ necessarily satisfies $\tilde{g}(\iota(\Lambda)) = \iota(\Lambda)$ and $\tilde{g}(v) = \pm v$, otherwise \tilde{g} cannot be an isometry.

The following Lemma is a special case of [Nik80, Cor. 1.5.2].

Lemma 4.3 Let Λ be the generalized Kummer or $K3^{[n]}$ lattice. Write $\Lambda = w^\perp \subset \tilde{\Lambda}$ with w primitive (cf. Remark 4.2). An isometry $g \in O(\Lambda)$ can be extended to an isometry $\tilde{g} \in O(\tilde{\Lambda})$ if and only if g acts as ± 1 on the discriminant Λ^\vee/Λ .

Proof: By [Nik80, Cor. 1.5.2] we can extend g to such a \tilde{g} if and only if we have an isometry $\varphi : \Lambda^\perp \rightarrow \Lambda^\perp$ with an additional property. Since $\Lambda^\perp = \langle w \rangle$ the only two isometries are $\varphi = \pm 1$. Following the exposition in [Nik80, 5. ff.], the additionally property for $\varphi = \pm 1$ means that g acts on Λ^\vee/Λ as ± 1 . \square

Corollary 4.4 Let $\Lambda = w^\perp \subset \tilde{\Lambda}$ be as in the Lemma above and let denote $[\iota] = O(\tilde{\Lambda})\iota$ an arbitrary invariant $O(\tilde{\Lambda})$ -orbit of primitive isometric embeddings $\Lambda \hookrightarrow \tilde{\Lambda}$. Then

the sub group $\mathcal{W}(\Lambda) \subset \mathrm{O}^+(\Lambda)$ defined in Definition 2.27 is equal to the sub group of all $g \in \mathrm{O}^+(\Lambda)$ leaving the orbit $[\iota] = \mathrm{O}(\tilde{\Lambda})\iota$ invariant, i.e. there exists \tilde{g} such that $\iota \circ g = \tilde{g} \circ \iota$.

Proof: An element $g \in \mathrm{O}^+(\Lambda)$ leaves $\mathrm{O}(\tilde{\Lambda})\iota$ invariant if and only if it acts by ± 1 on the discriminant Λ^\vee/Λ by Lemma 4.3. \square

In other words, $\mathcal{W}(\Lambda) = \mathrm{Stab}([\iota])$ is equal to the stabilizer of $[\iota]$ with respect to the action of $\mathrm{O}^+(\Lambda)$ on the set of $\mathrm{O}(\tilde{\Lambda})$ -orbits of primitive isometric embeddings $\mathrm{O}(\Lambda, \tilde{\Lambda})$.

With the knowledge of the monodromy group of a generalized Kummer manifold, see Theorem 2.4, one can construct an analogue of the monodromy invariant $\mathrm{O}(\tilde{\Lambda})$ -orbit as in [Mar10, Thm. 1.10].

Let S be an abelian surface and let $H^\bullet(S)$ denote the even cohomology i.e.

$$H^\bullet(S) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

together with the bilinear form defined by $(v, w) := (v_2, w_2) - \int_S (v_0 \wedge w_4 + v_4 \wedge w_0)$ where $(v_2, w_2) = \int_S v_2 \wedge w_2$ denotes the intersection form on $H^2(S, \mathbb{Z})$ and $v = v_0 + v_2 + v_4$ with $v_i \in H^i(S, \mathbb{Z})$ the decomposition in $H^\bullet(S)$ and similarly for w . This lattice is even, unimodular, of rank 8 and isometric to the *Mukai lattice*

$$(4.5) \quad \tilde{\Lambda} := \mathrm{U}^{\oplus 4}$$

where U is the unimodular rank two hyperbolic lattice. We identify $H^4(S, \mathbb{Z}) = \mathbb{Z}$ where we use the Poincare dual to a point as a generator and similarly $H^0(S, \mathbb{Z}) = \mathbb{Z}$ by taking the Poincare dual of S .

Definition 4.6 A *Mukai vector* is a tuple $v = (r, c, s)$ in $H^0(S, \mathbb{Z}) \oplus H^{1,1}(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$. It is called *positive* if one of the following cases are satisfied

- (i) $r > 0$
- (ii) $r = 0$, c is effective and $s \neq 0$
- (iii) $r = c = 0$ and $s < 0$

Let v be a primitive Mukai vector on S . An ample divisor H on S is called v -generic if every H -semistable sheaf is H -stable. For a coherent sheaf $F \in \mathrm{Coh}(S)$ set $v(F) := \mathrm{ch}(F)$ ² which is a Mukai vector as easily verified. Choose a positive and primitive Mukai vector $v = (r, c, s)$ with $c \in \mathrm{NS}(S)$ and $(v, v) \geq 6$ together with a v -generic ample class H . General results of S. Mukai [Muk84] imply that the moduli space $M_H(v)$ of H -stable sheaves F with Mukai vector $v(F) = v$ is a projective holomorphic symplectic manifold but not irreducible. By [Yos01, Thm. 0.1] the Albanese torus of $M_H(v)$ is $S \times S^\vee$. Consider the Albanese map

$$\mathrm{Alb}_v : M_H(v) \longrightarrow S \times S^\vee$$

and set $K_H(v) := \mathrm{Alb}_v^{-1}(0, 0)$. Then we have $\dim K_H(v) = (v, v) - 2 =: 2n$ and by K. Yoshioka [Yos01, Thm 0.2] this is an irreducible holomorphic symplectic manifold of Kummer type.

We have Mukai's homomorphism of Hodge structures

$$\Theta_v : v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z})$$

which can be defined as follows. Choose a quasi-universal family of sheaves \mathcal{E} on S of simplitude $\rho \in \mathbb{N}$, cf. [Muk87, Thm. A.5]. That is a family of sheaves $\mathcal{E} \in$

²Note that $v(F) = (\mathrm{rk}(F), c_1(F), c_1^2(F)/2 - c_2(F))$

$\text{Coh}(S \times M_H(v))$ on S parametrized by $M_H(v)$ (in particular, \mathcal{E} is flat over $M_H(v)$) and for every class $F \in M_H(v)$ one has $\mathcal{E}_{[F]} = \mathcal{E}|_{S \times \{F\}} \cong F^{\oplus \rho}$. Then set

$$(4.7) \quad \Theta_v(x) := \frac{1}{\rho} \left[(\text{pr}_{M_H(v)})^* \left((\text{ch}(\mathcal{E})(\text{pr}_S)^*(\sqrt{\text{Td}(S)}x^\vee) \right) \right]_2$$

where $x^\vee = -x_0 + x_2 + x_4$ for $x = x_0 + x_2 + x_4$ and $[\cdot]_2$ denotes the part in $H^2(S, \mathbb{Z})$. Note that $\sqrt{\text{Td}(S)} = 1$ for an abelian surface S . For the details see [Yos01, 1.2], [O'G97], [Muk87] and [Muk84].

By composing with the restriction mapping $r : H^2(M_H(v), \mathbb{Z}) \rightarrow H^2(K_H(v), \mathbb{Z})$ we obtain an morphism

$$(4.8) \quad \Theta_v : v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z}) \longrightarrow H^2(K_H(v), \mathbb{Z})$$

which is an isometry of Hodge structures by [Yos01, Thm. 0.2] and which we also denote by Θ_v by abuse of notation.

Theorem 4.1 *Let X be a manifold of generalized Kummer type of dimension $2n \geq 4$. Then there exists a canonical monodromy invariant $\text{O}(\tilde{\Lambda})$ -orbit ι_X of primitive isometric embeddings $\Lambda = H^2(X, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}$ into the Mukai lattice.*

Proof: Let $K_H(v)$ denote the manifold of generalized Kummer type described above such that $\dim X = \dim K_H(v)$. Fix an isometry $\varphi : H^\bullet(S) \rightarrow \tilde{\Lambda}$ and let $P : H^2(X, \mathbb{Z}) \rightarrow H^2(K_H(v), \mathbb{Z})$ be a parallel transport operator. Denote by ι the primitive isometric embedding

$$H^2(X, \mathbb{Z}) \xrightarrow{P} H^2(K_H(v), \mathbb{Z}) \xrightarrow{\Theta_v^{-1}} v^\perp \xrightarrow{\varphi} \tilde{\Lambda}.$$

Set $\iota_X := \text{O}(\tilde{\Lambda})\iota$. Let $g \in \text{Mon}^2(X)$ denote a monodromy operator. By Theorem 2.4 g acts on $H^2(X, \mathbb{Z})^\vee / H^2(X, \mathbb{Z})$ as $\pm \text{id}$. By Lemma 4.3 g can be extended to an isometry \tilde{g} of $\tilde{\Lambda}$ such that $\iota \circ g = \tilde{g} \circ \iota$, i.e. the orbit ι_X is monodromy invariant.

The orbit ι_X is *canonical* in the following sense. We have made a choice of moduli spaces $K_H(v) \subset M_H(v)$ of sheaves on an abelian surface S and therefore of Mukai's homomorphism $\Theta_v : v^\perp \rightarrow H^2(M_H(v), \mathbb{Z}) \rightarrow H^2(K_H(v), \mathbb{Z})$. It might be, that a different choice of moduli spaces and therefore of a different Mukai homomorphism could lead to another orbit of primitive isometric embeddings. With *canonical* we mean that we always end up with the same orbit.

This follows from K. Yoshioka's method of proof of the main results in [Yos01, 4.3., Prop. 4.12., Proof of Thm. 0.1 and 0.2]. If we choose another irreducible holomorphic symplectic moduli space of dimension $\dim X$, then it is deformation equivalent to $K_H(v)$ and Yoshioka's proof for this statement uses deformations of moduli spaces of sheaves over families of surfaces [Yos01, Lem. 2.3], and Fourier–Mukai transforms for which the Mukai homomorphism varies continuously, see [Yos01, 2.2., Proof of Prop. 2.4.]. Therefore the $\text{O}(\tilde{\Lambda})$ -orbit does not change. \square

5. Monodromy Invariants

We start with basic facts about general monodromy invariants, as described in [Mar13, 5.3.]. In the next subsection the monodromy invariant for isotropic classes for the generalized Kummer case is constructed.

Let X be an irreducible holomorphic symplectic manifold. Let $I(X) \subset H^2(X, \mathbb{Z})$ denote a monodromy invariant subset, i.e. $\text{Mon}^2(X) \cdot I(X) \subset I(X)$ and Σ a set.

Definition 5.1 [Mar13, Def. 5.16] A *monodromy invariant* of the pair (X, e) , $e \in I(X)$, is a $\text{Mon}^2(X)$ -invariant map $\vartheta : I(X) \rightarrow \Sigma$ i.e. $\vartheta(ge) = \vartheta(e)$ for all $e \in I(X)$ and all $g \in \text{Mon}^2(X)$. Further ϑ is called *faithful* if the induced map $\bar{\vartheta} : I(X)/\text{Mon}^2(X) \rightarrow \Sigma$ is injective.

5.2. Induced monodromy invariant subset. Let X' denote another irreducible holomorphic symplectic manifold deformation equivalent to X . Let $P : H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ denote a parallel transport operator. Then we can define

$$I(X') := P(I(X))$$

to obtain a $\text{Mon}^2(X')$ invariant subset $I(X')$ of $H^2(X', \mathbb{Z})$ induced by $I(X)$. Indeed this is well defined: if one has another parallel transport operator $P' : H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$, then $P'^{-1} \circ P$ is in $\text{Mon}^2(X)$ hence $(P'^{-1} \circ P)(I(X)) = I(X)$ as $I(X)$ is $\text{Mon}^2(X)$ invariant. Hence $P(I(X)) = P'(I(X))$.

Alternatively we could define

$$I(X') = \left\{ e' \in H^2(X', \mathbb{Z}) \mid \text{there exists } e \in I(X) \text{ such that } (X, e) \sim_{\text{def}} (X', e') \right\}.$$

where the deformation equivalence of the pairs (X, e) and (X', e') is meant in the sense of Definition 2.4 as usual.

5.3. Induced monodromy invariant. Let X' be as above. If we have a monodromy invariant $\vartheta : I(X) \rightarrow \Sigma$ then we can obtain an induced monodromy invariant on X' which we also denote by $\vartheta : I(X') \rightarrow \Sigma$ by abuse of notation. If $e' \in I(X')$ then there is a pair (X, e) deformation equivalent to (X', e') and we can define the induced monodromy invariant by

$$\vartheta(e') := \vartheta(e).$$

Note that this is well defined as ϑ is $\text{Mon}^2(X)$ -invariant.

The following is a very important statement for the computation of polarization types of Lagrangian fibrations and is based on the Global Torelli Theorem, see [Mar13, 5.2 ff.].

Proposition 5.4 [Mar13, Lem. 5.17] *Let $\vartheta : I(X) \rightarrow \Sigma$ be a faithful monodromy invariant and let (X_i, e_i) , $i = 1, 2$, denote two pairs with X_i deformation equivalent to X and $e_i \in I(X_i)$.*

- (i) $\vartheta(e_1) = \vartheta(e_2)$ if and only if (X_1, e_1) and (X_2, e_2) are deformation equivalent.
- (ii) If $\vartheta(e_1) = \vartheta(e_2)$ and $e_i = c_1(L_i)$ for holomorphic line bundles L_i on X_i and there exist Kähler classes ω_i on X_i such that $(\omega_i, e_i) > 0$, then (X_1, L_1) is deformation equivalent to (X_2, L_2) .

For effective isotropic classes, the requirements of the second statement of the Proposition above is always satisfied due to the following Lemma.

Lemma 5.5 [Wie16, Lem. 6.7] *Let λ be a nontrivial isotropic class in the closure $\bar{\mathcal{C}}_X$ of the positive cone in $H^{1,1}(X, \mathbb{R})$ with X an arbitrary irreducible holomorphic symplectic manifold. Then the Beauville–Bogomolov quadratic form satisfies $(x, \lambda) > 0$ for every class x in the positive cone \mathcal{C}_X .*

By definition the positive cone \mathcal{C}_X contains the Kähler cone \mathcal{K}_X , therefore we always find Kähler classes as required in (ii) of Proposition 5.4, if the considered classes e_i are isotropic.

5.6. Monodromy invariants for isotropic classes. As one expects a close relation between Lagrangian fibrations and isotropic line bundles, similar for K3 surfaces, we are interested in monodromy invariants defined on the subset of isotropic classes of the second cohomology of an irreducible holomorphic symplectic manifold. In this section a monodromy invariant for the isotropic classes on generalized Kummer manifolds is constructed in analogy of [Mar11, 2.]

Let X be a generalized Kummer type manifold of dimension $2n$. By Theorem 4.1 we have a canonical monodromy invariant $O(\tilde{\Lambda})$ -orbit ι_X of primitive isometric embeddings from $\Lambda := H^2(X, \mathbb{Z})$ into the Mukai lattice $\tilde{\Lambda}$ (4.5). Choose the following:

- (i) A representative $\iota : \Lambda \hookrightarrow \tilde{\Lambda}$ in ι_X .
- (ii) A generator v of the sublattice $\iota(\Lambda)^\perp = \langle v \rangle$, cf. Remark 4.2.

Remark 5.7 The Kummer type lattice Λ has signature $(3, 4)$, hence the orthogonal complement $\iota(\Lambda)^\perp$ is positive definite of rank one as the Mukai lattice $\tilde{\Lambda} = U^{\oplus 4}$ has signature $(4, 4)$. Since the Gram discriminant of Λ is $-(2n+2)$ the Gram discriminant of $\iota(\Lambda)^\perp$ is $2n+2$, hence $(v, v) = 2n+2$. Furthermore, by [Nik80, Thm 1.14.4] there is a unique orbit of such primitive elements with square $2n+2$ (respectively $2n-2$ in the $K3^{[n]}$ case) in $\tilde{\Lambda}$. Since $\iota(\Lambda) = v^\perp$ we conclude that the action of $O(\Lambda) \times O(\tilde{\Lambda})$ on $O(\Lambda, \tilde{\Lambda})$ is transitive.

For a primitive and isotropic element α in the Kummer type lattice Λ denote by $H(\alpha, \iota)$ the lattice defined by

$$(5.8) \quad H(\alpha, \iota) := \text{sat} \langle \iota(\alpha), v \rangle = \text{sat} \langle \iota(\alpha), -v \rangle ,$$

where sat denotes the saturation – the *saturation* of a sublattice L is the maximal sublattice of the same rank containing L .

Definition 5.9 (i) Let Λ_1, Λ_2 denote lattices and $e_i \in \Lambda_i$ elements. A *morphism of the pairs* (Λ_i, e_i) is an isometry $g : \Lambda_1 \rightarrow \Lambda_2$ such that $g(e_1) = e_2$.
(ii) The *divisibility* or the *divisor* of an element $x \in \Lambda$ is defined as

$$\text{Div}(x) := \max \{k \in \mathbb{N} \mid (x, \cdot)/k \text{ is an integral class in the dual } \Lambda^\vee\} .$$

Equivalently, $\text{Div}(x)$ is the unique positive generator of the ideal $(x, L) = \text{Div}(x)\mathbb{Z} \subset \mathbb{Z}$. Note that if the lattice is unimodular, then $\text{Div}(x) = 1$ for every primitive element x .

Further denote by

$$(5.10) \quad \vartheta(\alpha) := [(H(\alpha, \iota), v)]$$

the isometry class of the pair $(H(\alpha, \iota), v)$.

Let d be a positive number such that d^2 divides $2n+2$. Then define the lattice $L_{n,d}$ as \mathbb{Z}^2 with form

$$(5.11) \quad \frac{2n+2}{d^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

The following Lemma is very similar to [Mar14, Lem. 2.5].

Lemma 5.12 Let $\alpha \in \Lambda$ be a primitive isotropic class and set $d := \text{Div}(\alpha)$.

- (i) $\vartheta(\alpha)$ does not depend on the chosen representative $\iota \in \iota_X$.
- (ii) For all $g \in \text{Mon}^2(X)$ we have $\vartheta(g(\alpha)) = \vartheta(\alpha)$.

(iii) We can compose $\alpha \in \Lambda \cong U^{\oplus 3} \oplus \langle -2n-2 \rangle$ as

$$\alpha = d\xi + b\delta$$

where $\xi \in U^{\oplus 3}$ is primitive, δ is the generator of $\langle -2n-2 \rangle$ and $\gcd(d, b) = 1$. Further d^2 divides $n+1$.

(iv) The lattice $H(\alpha, \iota)$ is isometric to the lattice $L_{n,d}$ defined in (5.11).

(v) There is an integer b , namely the one in (iii), such that $(\iota(\alpha) - bv)/d$ is integral (i.e. contained in $H(\alpha, \iota)$). Also any integer b with

- $\gcd(d, b) = 1$ and
- $(\iota(\alpha) - bv)/d$ is integral

satisfies $\vartheta(\alpha) = [(L_{n,d}, (d, b))]$.

Proof:

(i) Let $\iota_i \in \iota_X$, $i = 1, 2$, be two representatives with $\iota_i(\Lambda)^\perp = \langle v_i \rangle$. Since the ι_i are in the same orbit ι_X there exists $\tilde{g} \in O(\tilde{\Lambda})$ such that $\tilde{g} \circ \iota_1 = \iota_2$ hence $\tilde{g}(\iota_1(\Lambda)) = \iota_2(\Lambda)$. We necessarily have $\tilde{g}(v_1) = \pm v_2$, otherwise we would have a contradiction to the bijectivity of \tilde{g} . We can assume $\tilde{g}(v_1) = v_2$ (otherwise take $-\tilde{g}$) then $\tilde{g}(\langle \iota_1(\alpha), v_1 \rangle) = \langle \iota_2(\alpha), v_2 \rangle$ and the same holds for the saturation. Consequently \tilde{g} gives the desired isometry of the pairs $(H(\alpha, \iota_i), v_i)$ hence $\vartheta(\alpha)$ does not depend on the chosen ι .

(ii) The orbit $\iota_X = O(\tilde{\Lambda})\iota$ is monodromy invariant that means we have a $\tilde{g} \in O(\tilde{\Lambda})$ such that $\tilde{g} \circ \iota = \iota \circ g$. With the same argument as in (i) we have $\tilde{g}(v) = \pm v$ (see Remark 4.2) and can assume $\tilde{g}(v) = v$. So \tilde{g} defines an isometry between $\langle \iota(\alpha), v \rangle$ and $\langle \iota(g(\alpha)), v \rangle$ since $\tilde{g}(\iota(\alpha)) = \iota(g(\alpha))$ and in particular an isometry between the saturations $(H(\alpha, \iota), v)$ and $(H(g(\alpha), \iota), v)$, hence $\vartheta(\alpha) = \vartheta(g(\alpha))$.

(iii) Let δ be the generator of $\langle -2n-2 \rangle \subset \Lambda$. Then $\delta_\Lambda^\perp = U^{\oplus 3}$. Since α is primitive we can write $\alpha = a\xi + b\delta$ such that $a > 0$ and $\xi \in \delta^\perp = U^{\oplus 3}$ and $\gcd(a, b) = 1$. Then

$$0 = (\alpha, \alpha) = a^2(\xi, \xi) - (2n+2)b^2 \Leftrightarrow a^2(\xi, \xi) = (2n+2)b^2.$$

As (ξ, ξ) is even we get that a^2 divides $(n+1)$. Since δ is primitive we have $\text{Div}(\delta) = 2n+2$ and $\text{Div}(\xi) = 1$ since ξ is primitive and $U^{\oplus 3}$ is unimodular, hence

$$d = \text{Div}(\alpha) = \gcd(\text{Div}(a\xi), \text{Div}(b\delta)) = \gcd(a, (2n+2)b) = a.$$

(iv),(v) We use the same notation as in (iii). The lattice $\iota(U^{\oplus 3})^\perp \subset \tilde{\Lambda}$ is of rank 2 and contains $\iota(\delta)$ and v , hence it is the saturation of $\langle \iota(\delta), v \rangle$ as orthogonal complements are always saturated. As a complement of a unimodular lattice it is unimodular itself, hence it is the hyperbolic plane U . Consequently we can assume that $v = (1, n+1)$ and $\iota(\delta) = (1, -n-1)$. We have $\iota(\delta) - v = (2n+2)e$ where $e = (0, -1)$. Clearly e is isotropic. Then set

$$u := \frac{1}{d}(bv - \iota(\alpha)) = -\iota(\xi) - \frac{b}{d}(2n+2)e.$$

Hence, the existence of such an integer b is proven.

As $\iota(\alpha) = -du + bv$ we have $\langle v, u \rangle \subset H(\alpha, \iota) := \text{sat} \langle \iota(\alpha), v \rangle$. The complement $\delta_\Lambda^\perp = U^{\oplus 3}$ is unimodular, hence we can find $\eta \in \delta_\Lambda^\perp$ such that $(\eta, \xi) = 1$ as $\xi \in U^{\oplus 3}$ is primitive. For the intersection numbers we have

$$\begin{pmatrix} (v, e) & (v, \iota(\eta)) \\ (u, e) & (u, \iota(\eta)) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore the sublattice $\langle v, u \rangle \subset \tilde{\Lambda}$ must be saturated, otherwise the determinant of the matrix above must be divisible by a nontrivial square. Consequently we have $H(\alpha, \iota) := \text{sat } \langle \iota(\alpha), v \rangle = \langle v, u \rangle$.

Further $(v, u) = b \frac{2n+2}{d}$ and $(u, u) = b^2 \frac{2n+2}{d^2}$. The Gram matrix G of $H(\alpha, \iota)$ with respect to the basis v, u is therefore

$$G = \frac{2n+2}{d^2} \begin{pmatrix} d^2 & bd \\ bd & b^2 \end{pmatrix} = \frac{2n+2}{d^2} \begin{pmatrix} d \\ b \end{pmatrix} \begin{pmatrix} d & b \end{pmatrix}.$$

Since $\gcd(d, b) = 1$ there are integers $i, j \in \mathbb{Z}$ with $id + jb = 1$. Set

$$A := \begin{pmatrix} i & j \\ b & -d \end{pmatrix}.$$

This is an integral matrix with $A(d, b)^t = (1, 0)^t$ and determinant -1 , hence invertible over the integers. The Gram matrix with respect to the base change A is

$$A^t G A = \frac{2n+2}{d^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore we have an isomorphism $L_{n,d} \cong H(\alpha, \iota)$ of lattices via $(x, y)^t \mapsto A(x, y)^t \cdot (v, u)$ where the product \cdot is seen as a formal euclidean product. In particular (d, b) is mapped to v i.e. $\vartheta(\alpha) = [(L_{n,d}, (d, b))]$.

Now let b' be any integer satisfying the assumptions in (v). We know that (d, b') is primitive and that

$$u' := \frac{1}{d}(b'v - \iota(\alpha))$$

is integral, therefore $u' - u = \frac{b'-b}{d}v$ is also integral. Since v is primitive, d must divide $b' - b$. Set $c := \frac{b'-b}{d} \in \mathbb{Z}$. Then

$$g_c := \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \text{O}(L_{n,d})$$

is clearly an isometry of $L_{n,d}$ with $g_c(d, b)^t = (d, dc + b)^t = (d, b')^t$. Hence $\vartheta(\alpha) = [L_{n,d}, (d, b)] = [L_{n,d}, (d, b')]$. □

Recall the following useful and well known criterion of M. Eichler.

Proposition 5.13 (EICHLER'S CRITERION, [Eic52, 10.]) *Let Λ be an even lattice which contains two copies of the hyperbolic plane U . Then the $\text{O}(\Lambda)$ -orbit of a primitive element $x \in \Lambda$ is determined by its length (x, x) and the element $(x, \cdot)/\text{Div}(x) \in \Lambda^\vee$.*

Lemma 5.14 *The degenerate lattice $L_{n,d}$ embeds primitively and isometrically into $\tilde{\Lambda} = U^{\oplus 4}$ uniquely up to an isometry in $\text{O}(\tilde{\Lambda})$.*

Proof: Write $L := L_{n,d}$. The existence of such an embedding is clear, but follows also from [Nik80, Prop. 1.17.1]: there exists a primitive isometric embedding $L \hookrightarrow \tilde{\Lambda}$ if and only if we can embed the quotient $L/\ker L$, where $\ker L$ denotes the null space of L , into some lattice of signature $(4-r, 4-r)$ where $r = \text{rk } \ker L$. Since $\ker L = \langle (0, 1) \rangle$ and $L/\ker L \cong \left\langle \frac{2n+2}{d^2} \right\rangle$, this is clearly possible.

Also by [Nik80, Prop. 1.17.1] the isomorphism classes of primitive isometric embeddings $j : L \hookrightarrow \tilde{\Lambda}$ are in one to one correspondence with isomorphism classes of induced primitive isometric embeddings

$$L/\ker L \hookrightarrow (\ker L)_{\tilde{\Lambda}}^{\perp}/\ker L.$$

By Eichler's criterion 5.13, we can assume that $j(0, 1) = ((0, 1), 0, 0, 0)$ and $j(1, 0) = (0, (\frac{n+1}{d^2}, 1), 0, 0)$. Then $(\ker L)_{\tilde{\Lambda}}^{\perp} = \langle 0 \rangle \oplus U^{\oplus 3}$, hence $(\ker L)_{\tilde{\Lambda}}^{\perp}/\ker L \cong U^{\oplus 3}$. Now by Eichler's criterion 5.13 there is up to an isometry in $O(\tilde{\Lambda})$ a unique way to embed $L/\ker L = \langle \frac{2n+2}{d^2} \rangle$ into $U^{\oplus 3}$. \square

Lemma 5.15 *Let $\alpha \in \Lambda = U^{\oplus 3} \oplus \langle -2n-2 \rangle$ be a primitive isotropic element in the Kummer type lattice. Then there exists a $u \in \Lambda$ such that $(u, \alpha) = 0$ and $(u, u) = \pm 2$.*

Proof: Write $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 \in U^{\oplus 3}$ and $\alpha \in \langle -2n-2 \rangle$. The discriminant of $U^{\oplus 3}$ is trivial since it's unimodular, hence by Eichler's criterion 5.13 the $O(U^{\oplus 3})$ -orbit of α_0 is determined by it's length $(\alpha_0, \alpha_0) = 2n+2$. So there exists an isometry $g \in O(U^{\oplus 3})$ such that $g(\alpha_0) = ((1, n+1), 0, 0) \in U^{\oplus 3}$. Set $u := g^{-1}(0, 0, (1, \pm 1)) \in U^{\oplus 3} \subset \Lambda$. Then $(u, \alpha) = (u, \alpha_0) = 0$ and $(u, u) = ((1, \pm 1), (1, \pm 1)) = \pm 2$. \square

For a positive integer d let $I_d(X) \subset \Lambda = H^2(X, \mathbb{Z})$ denote the subset of primitive isotropic elements α such that $\text{Div}(\alpha) = d$ which is clearly a $\text{Mon}^2(X)$ -invariant subset. Let $\Sigma_{n,d}$ denote the set of isometry classes of pairs (H, w) such that H is isometric to $L_{n,d}$ and $w \in H$ is a primitive class with $(w, w) = 2n+2$.

Theorem 5.1 *Let X be a Kummer type manifold of dimension $2n$ and d a positive integer such that d^2 divides $n+1$. The mapping*

$$\vartheta : I_d(X) \longrightarrow \Sigma_{n,d}, \quad \alpha \longmapsto \vartheta(\alpha) = [(H(\alpha, \iota), v)]$$

is a surjective faithful monodromy invariant of the manifold X .

Proof: By Lemma 5.12 $\vartheta : I_d(X) \longrightarrow \Sigma_{n,d}$ is well defined and $\text{Mon}^2(X)$ -invariant.

To show that ϑ is faithful i.e. that the induced map $\vartheta : I_d(X)/\text{Mon}^2(X) \longrightarrow \Sigma_{n,d}$ is injective, we assume $\alpha_1, \alpha_2 \in I_d(X)$ with $\vartheta(\alpha_1) = \vartheta(\alpha_2)$, that means we have an isometry $g : H(\alpha_1, \iota) \rightarrow H(\alpha_2, \iota)$ with $g(v) = v$ where v is as usual a generator of $\iota(\Lambda)^{\perp}$.

We first show that both α_i lie in the same $\mathcal{W}(\Lambda)$ -orbit, where the group $\mathcal{W}(\Lambda)$ was defined in Definition 2.27. We have $H(\alpha_i, \iota) \cong L_{n,d}$. By Lemma 5.14 there is up to an isometry in $O(\tilde{\Lambda})$ a unique way to embed $H(\alpha_i, \iota)$ isometrically and primitively into $\tilde{\Lambda}$, hence we can extend g to an isometry $\tilde{g} \in O(\tilde{\Lambda})$. Since $v^{\perp} = \iota(\Lambda)$ we have in particular $\tilde{g}(\iota(\Lambda)) = \iota(\Lambda)$, i.e it makes sense to set $h := \iota^{-1} \circ \tilde{g} \circ \iota$ which is an isometry $h \in O(\Lambda)$ such that $\iota \circ h = \tilde{g} \circ \iota$, hence h leaves the orbit $\iota_X = O(\tilde{\Lambda})\iota$ invariant and by Lemma 4.4 either $\mu = h$ or $\mu = -h$ is contained in the subgroup $\mathcal{W}(\Lambda)$ of orientation preserving isometries acting as ± 1 on the discriminant Λ^{\vee}/Λ . Choose μ such that it is in $\mathcal{W}(\Lambda)$. The null space of $H(\alpha_i, \iota) \subset \tilde{\Lambda}$ is generated by $\iota(\alpha_i)$. Since $\tilde{g} \in O(\tilde{\Lambda})$ restricts to an isometry between $H(\alpha_1, \iota)$ and $H(\alpha_2, \iota)$ the null space of $H(\alpha_2, \iota)$ is generated by $\tilde{g}(\iota(\alpha_1)) = \iota(\pm h(\alpha_1)) = \iota(\mu(\alpha_1))$. So we have $\iota(\mu(\alpha_1)) = \pm \iota(\alpha_2)$, hence $\mu(\alpha_1) = \pm \alpha_2$. By Lemma 5.15 we can choose a $u \in \Lambda$ with $(u, \alpha_2) = 0$ and $(u, u) = +2$. Then the isometry $\rho_u \in O(\Lambda)$ defined in Definition 2.29

i.e. $\rho_u(x) = -R_u(x) = -x + (u, x)u$ is contained in $\mathcal{W}(\Lambda)$, see Corollary 2.31 and Remark 2.30, and satisfies $\rho_u(\alpha_2) = -\alpha_2$, hence

$$\mathcal{W}(\Lambda)\alpha_1 = \mathcal{W}(\Lambda)(\pm\alpha_2) = \mathcal{W}(\Lambda)\alpha_2.$$

Now we show that $\mathcal{W}(\Lambda)\alpha = \text{Mon}^2(X)\alpha$ for every primitive isotropic element $\alpha \in \Lambda$. Since $\text{Mon}^2(X) \subset \mathcal{W}(\Lambda)$ is an index 2 subgroup by Corollary 2.31 we can write

$$\mathcal{W}(\Lambda) = \text{Mon}^2(X) \cup \text{Mon}^2(X)w$$

for every $w \in \mathcal{W}(\Lambda) \setminus \text{Mon}^2(X)$. By Lemma 5.15 we have an element $u \in \Lambda$ with $(u, \alpha) = 0$ and $(u, u) = -2$. Then the reflection $\rho_u(x) = R_u(x) = x + (u, x)u$ of Λ (Definition 2.29) acts as $+1$ on the discriminant but has determinant -1 , hence it is contained in $\mathcal{W}(\Lambda)$ but not in $\text{Mon}^2(X)$, see again Corollary 2.31 and Remark 2.30. In particular $\rho_u(\alpha) = \alpha$ therefore

$$\begin{aligned} \mathcal{W}(\Lambda)\alpha &= (\text{Mon}^2(X) \cup \text{Mon}^2(X)\rho_u)\alpha \\ &= \text{Mon}^2(X)\alpha \cup \text{Mon}^2(X)\rho_u(\alpha) = \text{Mon}^2(X)\alpha. \end{aligned}$$

For surjectivity, assume we have a class $[(L_{n,d}, w)] \in \Sigma_{n,d}$, i.e. $w \in L_{n,d}$ is primitive such that $(w, w) = 2n + 2$. By Lemma 5.14 there exists a primitive isometric embedding $\iota_{n,d} : L_{n,d} \hookrightarrow \tilde{\Lambda}$.

By Eichler's criterion 5.13 we can assume that $\iota_{n,d}(w)$ is contained in a copy of \mathbf{U} of $\tilde{\Lambda} = \mathbf{U}^{\oplus 4}$. Then the lattice $\iota_{n,d}(w)^\perp \subset \tilde{\Lambda} = \mathbf{U}^{\oplus 4}$ is of signature $(3, 4)$ and since $(w, w) = 2n + 2$ the complement $\iota_{n,d}(w)^\perp$ is isomorphic to $\Lambda \cong \mathbf{U}^{\oplus 3} \oplus \langle -2n - 2 \rangle$.

The action of $\text{O}(\Lambda) \times \text{O}(\tilde{\Lambda})$ on $\text{O}(\Lambda, \tilde{\Lambda})$ is transitive by Remark 5.7, hence the induced action of $\text{O}(\Lambda)$ on the orbit set $\text{O}(\Lambda, \tilde{\Lambda})/\text{O}(\tilde{\Lambda})$ is also transitive. Hence, we can choose an isometry $g : \iota_{n,d}(w)^\perp \rightarrow \Lambda$ such that

$$\kappa : \Lambda \xrightarrow{g^{-1}} \iota_{n,d}(w)^\perp \subset \tilde{\Lambda}$$

belongs to the monodromy invariant orbit $\iota_X = \text{O}(\tilde{\Lambda})\iota$. Recall from above that $(0, 1) \in \ker L_{n,d}$ is the generator of $\ker L_{n,d}$. Clearly we have $(w, (0, 1)) = 0$ in $L_{n,d}$ so we can set $\alpha := g(\iota_{n,d}(0, 1))$. We can write

$$\alpha = a\xi + b\delta$$

where $\xi \in \mathbf{U}^{\oplus 3}$, $\delta \in \langle -2n - 2 \rangle$, $a > 0$ such that $\gcd(a, b) = 1$. As in the proof of Lemma 5.12 (iv) it follows that $a = \text{Div}(\alpha)$. We have $\kappa(\Lambda)^\perp = \langle \iota_{n,d}(w) \rangle$ and $\kappa(\alpha) = \iota_{n,d}((0, 1))$ and from Lemma 5.12 again

$$H(\alpha, \kappa) = \text{sat} \langle \iota_{n,d}(0, 1), \iota_{n,d}(w) \rangle \cong L_{n,a},$$

where $\iota_{n,d}(w)$ is mapped to (a, b) . The primitive element $w \in L_{n,d}$ is necessarily of the form $(\pm d, w_2)$ with $\gcd(d, w_2) = 1$. Over the rational numbers we have clearly $\iota_{n,d}(L_{n,d})_{\mathbb{Q}} = \langle \iota_{n,d}(0, 1), \iota_{n,d}(w) \rangle_{\mathbb{Q}}$. As $\iota_{n,d}(L_{n,d})$ is saturated it follows that

$$\iota_{n,d}(L_{n,d}) = \text{sat} \langle \iota_{n,d}(0, 1), \iota_{n,d}(w) \rangle = H(\alpha, \kappa).$$

Now we have an isometry

$$L_{n,d} \xrightarrow{\iota_{n,d}} H(\alpha, \kappa) \rightarrow L_{n,a}$$

where w is mapped to (a, b) , hence $\text{Div}(\alpha) = a = d$ i.e. $\alpha \in I_d(X)$ and $\vartheta(\alpha) = [(L_{n,d}, w)]$. \square

6. Beauville–Mukai systems of generalized Kummer type

We define the notion of a *Beauville–Mukai system of generalized Kummer type*. It is similar defined as in the $K3^{[n]}$ case, see [Wie16]. The fibers of them are not Jacobian of curves anymore, but an abelian subvariety of a Jacobian. Therefore we dwell on some theory of complementary subvarieties in Jacobians, see subsection 6.4.

Let S be an abelian surface and v be a primitive Mukai vector on S of the form $v = (0, c_1(D), s)$ where D is a ample divisor on S i.e. D is ample. We set $2n := (D, D) - 2$. Note that we have $h^0(S, D) = \frac{1}{2}(D, D) = n + 1$. Choose a v -generic ample class H on S , hence $M_H(v)$ is an holomorphic symplectic manifold as explained in section 4.

For simplicity we now fix a reference point $F_0 \in M_H(v)$ such that $\det(F_0) = \mathcal{O}_S(D)$. By [Yos01] the Albanese map $\text{Alb}_v : M_H(v) \rightarrow S \times S^\vee$ with respect the reference point $F_0 \in M_H(v)$ can be written as

$$(\text{Alb}_v)_{F_0} = \alpha \times \det_{F_0}$$

where $\det_{F_0} : M_H(v) \rightarrow \text{Pic}^0(S) = S^\vee$ is defined as $\det_{F_0}(F) := \det(F) \otimes (\det(F_0))^{-1}$ and α can be defined as

$$(6.1) \quad \alpha(F) := \sum c_2(F) := \sum_i n_i x_i$$

where we view $c_2(F)$ in the Chow ring represented by the cycle $[\sum_i n_i x_i]$, see [Yos01, 4.1 ff.] and [O’G14, p. 11].

The Albanese fiber $K_H(v) = (\text{Alb}_v)_{F_0}^{-1}(0, 0)$ is an irreducible holomorphic symplectic manifold of dimension $2n$ see section 4 and for $F \in K_H(v)$ the fitting support $\text{supp}(F)$ is an element of the linear system $|D|$. This leads to the following commutative diagram

$$(6.2) \quad \begin{array}{ccccc} K_H(v) & \hookrightarrow & M_H(v) & \xrightarrow{(\text{Alb}_v)_{F_0}} & S \times S^\vee \\ \pi \downarrow & & \pi \downarrow & & \downarrow \text{pr}_{S^\vee} \\ |D| & \hookrightarrow & \{D\} & \longrightarrow & S^\vee = \text{Pic}^0(S) \end{array}$$

where $\{D\} \rightarrow S^\vee$ is the map $C \mapsto \mathcal{O}_S(C) \otimes \det(F_0)^{-1}$. The induced map $K_H(v) \rightarrow |D|$ is a Lagrangian fibration by Matsushita’s Theorem 2.2.

Definition 6.3 In the setting as above, the Lagrangian fibration

$$\pi : K_H(v) \longrightarrow |D|, \quad F \longmapsto \text{supp}(F)$$

is called a *Beauville–Mukai system of generalized Kummer type*.

6.4. An excursion to the theory of Jacobians. To consider the fibers of Beauville–Mukai systems of generalized Kummer type and polarizations on them, we deal with some theory of complementary abelian subvarieties.

If M is an abelian variety and $A \subset M$ is an abelian subvariety and L a polarization on M , then one can define a so called *complementary subvariety* B to A (with respect to L). We only consider the case when $L = \Theta$ is a principal polarization [BL03, 12.1], for the more general setting see [BL03, 5.3]. We denote the induced isogeny of L by ϕ_L .

We assume for this section, that Θ is a principal polarization, therefore we can identify M with its dual M^\vee via the homomorphism ϕ_Θ . By [BL03, Prop. 1.2.6]

for any polarization L the isogeny ϕ_L has always a \mathbb{Q} -inverse and we can define the \mathbb{Q} -endomorphism

$$g_A := \iota \circ \phi_{\iota^* L}^{-1} \circ \iota^\vee : M \otimes \mathbb{Q} \longrightarrow M \otimes \mathbb{Q}$$

where $\iota = \iota_A : A \hookrightarrow M$ denotes the inclusion. Choose a positive number m such that mg_A is an endomorphism of M . By [BL03, Prop. 12.1.3] we have

$$(6.5) \quad B := (\ker(mg_A))_0 \subset M = \ker \iota^\vee \cong (A/B)^\vee,$$

where $(\ker(mg_A))_0$ denotes the identity component. Further B is an abelian subvariety of M called the *complementary subvariety* to A (with respect to L). Conversely, A is also the complementary subvariety to B and (A, B) is called a *pair of complementary subvarieties* in M .

Proposition 6.6 [BL03, Cor. 12.1.5] *Let (A, B) be a pair of complementary abelian subvarieties in a principally polarized abelian variety (M, Θ) with $\dim A \geq \dim B = r$. Denote by ι_A and ι_B the inclusions of A and B into M respectively and assume $\underline{d}(\iota_B^* \Theta) = (d_1, \dots, d_r)$. Then $\underline{d}(\iota_A^* \Theta) = (1, \dots, 1, d_1, \dots, d_r)$.*

6.7. The case of a Jacobian. Let $\iota : C \hookrightarrow S$ be a smooth curve in an abelian surface S . Denote by Θ the principal polarization of the Jacobian $\text{Jac}(C)$ of C and define $K(C) := \ker \text{Jac}(\iota)$ to be the kernel of the homomorphism $\text{Jac}(\iota)$ induced by the inclusion $\iota : C \hookrightarrow S$ and by the universal property of the Jacobian [BL03, 11.4.1.], i.e. we have an exact sequence

$$K(C) \hookrightarrow \text{Jac}(C) \xrightarrow{\text{Jac}(\iota)} S.$$

Using the several identifications of the dual and double dual, the dual of the pullback or the double pullback

$$(\iota^*)^\vee = (\iota^*)^* : \text{Jac}(C) = (\text{Jac}(C))^\vee = \text{Pic}^0(C) \longrightarrow S = (S^\vee)^\vee = \text{Pic}^0(S^\vee)$$

is nothing but the map $\text{Jac}(\iota)$. Of course, you can also see $\text{Jac}(\iota)$ as the Albanese map induced by ι

$$\text{Alb}(\iota) : \text{Alb}(C) \longrightarrow \text{Alb}(S) = S$$

if you identify $\text{Alb}(C)$ and $\text{Jac}(C)$, cf. [BL03, Prop. 11.11.6]. More concretely, the map $\text{Jac}(\iota)$ viewed as a map $\text{Pic}^0(C) \rightarrow S$ is

$$(6.8) \quad \mathcal{O}_C \left(\sum n_i x_i \right) \mapsto \sum n_i x_i.$$

Indeed, for a given point c , denote by $\alpha_c : C \hookrightarrow \text{Jac}(C)$, $x \mapsto \mathcal{O}_C(x - c)$ the Abel-Jacobi map. Then

$$\text{Jac}(\iota)(\alpha_c(x)) = \text{Jac}(\iota)(\mathcal{O}_C(x - c)) = x - c = t_{-c}(x),$$

therefore it satisfies exactly the property of the unique morphism as described in [BL03, 11.4.1.].

We can see the dual S^\vee as an abelian subvariety of $\text{Jac}(C)$ in the following sense.

Lemma 6.9 *The pullback morphism $\iota^* : S^\vee \rightarrow \text{Jac}(C)$ is an injection. Therefore $K(C)$ is connected.*

Proof: Let L be a line bundle on S with $L|_C = \mathcal{O}_C$. We have the standard exact sequence

$$(6.10) \quad 0 \longrightarrow L \otimes \mathcal{O}_S(-C) \longrightarrow L \longrightarrow L|_C = \mathcal{O}_C \longrightarrow 0.$$

Since C is effective, the line bundle $L(-C) = L \otimes \mathcal{O}_S(-C)$ has no holomorphic sections i.e. $H^0(S, L(-C)) = 0$. In particular, $L(-C)$ cannot be ample (cf. [BL03, Prop. 4.5.2]), therefore the associated hermitian form of $c_1(L(-C))$ must have less

then four positive eigenvalues. By [BL03, Lem. 3.5.1] we then have $H^1(S, L^\vee(-C)) = 0$. The long exact sequence of (6.10) shows that $h^0(S, L) = h^0(S, \mathcal{O}_C) = 1$ i.e. L has a holomorphic section s . Since $0 = c_1(L|_C) = [V(s)]$, the zero set $V(s)$ of s is empty i.e. $L = \mathcal{O}_S(V(s)) = \mathcal{O}_S(0) = \mathcal{O}_S$.

For the second statement identify $\text{Jac}(C)^\vee = \text{Jac}(C)$ via the principal polarization. We have the short exact sequence

$$0 \longrightarrow S^\vee \longrightarrow \text{Jac}(C) \longrightarrow \text{Jac}(C)/S^\vee \longrightarrow 0$$

and by [BL03, Prop. 2.4.2] the dual sequence

$$0 \longrightarrow (\text{Jac}(C)/S^\vee)^\vee \longrightarrow \text{Jac}(C) \longrightarrow S \longrightarrow 0$$

is also exact. Hence $K(C) = \ker(\text{Jac}(C) \rightarrow S) \cong (\text{Jac}(C)/S^\vee)^\vee$ i.e. $K(C)$ is connected. \square

In other words we have the following.

Lemma 6.11 *The abelian subvarieties $K(C)$ and $S^\vee \xrightarrow{\iota^*} \text{Jac}(C)$ are a pair complementary abelian subvarieties of $\text{Jac}(C)$.*

Proof: With the discussion above we have $K(C) = \ker(\iota^*)^\vee$ which is exactly the definition as in (6.5) \square

We are interested in the type of the polarization induced by Θ .

Lemma 6.12 *Let L be a polarization on an abelian surface S of type $\underline{d}(L) = (d_1, d_2)$. Then $h^0(S, L) = d_1 d_2$. If $C \in |L|$ is a not necessarily smooth curve, then we have for its arithmetic genus $g_a = d_1 d_2 + 1$. Further, if $c_1(L)$ is primitive and $(L, L) = 2d$, then $\underline{d}(L) = (1, d)$.*

Proof: By the well known formula for the (arithmetic) genus, we have $g_a = 1 + \frac{1}{2}(C, C)$. By the geometric Riemann–Roch [BL03, 3.6 ff.] and since L is ample, we have

$$d_1 d_2 = \chi(L) = h^0(S, L) = \frac{1}{2}(C, C) = g_a - 1.$$

If $(L, L) = 2d$ and $c_1(L)$ is primitive, the equation above also shows $2d = (L, L) = 2d_1 d_2$. Since d_1 divides d_2 and (d_1, d_2) is primitive as $c_1(L)$ is primitive, we have $d_1 = 1$ i.e. $\underline{d}(L) = (1, d)$. \square

Remark 6.13 If L is a polarization on an abelian variety S with $\underline{d}(L) = (d_1, \dots, d_n)$, then by [BL03, 14.4] there is a natural polarization L_δ on the dual S^\vee , called *dual polarization*, characterized by the following equivalent properties

- (a) $\phi_L^* L_\delta$ is algebraically equivalent to $L^{d_1 d_n}$, (b) $\phi_{L_\delta} \phi_L = d_1 d_n \text{id}_S$.

Further the type is given by $\underline{d}(L_\delta) = (d_1, \frac{d_1 d_n}{d_{n-1}}, \dots, \frac{d_1 d_n}{d_2}, d_n)$. If we are on an abelian surface, then obviously $\underline{d}(L) = \underline{d}(L_\delta)$.

Lemma 6.14 *Let (S, L) denote a polarized abelian surface of type $\underline{d}(L) = (d_1, d_2)$. Then for every smooth curve $\iota : C \hookrightarrow S$ with $C \in |L|$ we have that the restriction*

$$\Theta|_{S^\vee} := (\iota^*)^* \Theta$$

is a polarization of type (d_1, d_2) , where Θ denotes the principal polarization on $\text{Jac}(C) = \text{Pic}^0(C)$ and $(\iota^)^*$ is viewed as a map $\text{Pic}(\text{Jac}(C)) \rightarrow \text{Pic}(S^\vee)$. In particular, if the Picard number is $\rho(S) = 1$, then $\Theta|_{S^\vee} = L_\delta$ where the latter is the dual polarization on S^\vee to L , cf. Remark 6.13.*

Proof: The proof is divided in three steps. In the first, we assume for the Picard number $\rho(S) = 1$ and show the existence of such a curve in $|L|$. In the second and still under the assumption $\rho(S) = 1$ it is shown that it holds for every smooth curve in $|L|$. In the third step we drop the restriction on the Picard number. We set $d := d_1 d_2$.

- We first assume $\rho(S) = 1$ for the Picard number and prove the existence of such a curve $C \in |L|$. Since $\rho(S) = 1$ we have also $\rho(S^\vee) = 1$. Note that we have $\underline{d}(L_\delta) = (d_1, d_2)$ by Remark 6.13.

Consider the isogeny $\phi_L : S \rightarrow S^\vee$. Then

$$(6.15) \quad \ker \phi_L \cong (\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_1\mathbb{Z}) \oplus (\mathbb{Z}/d_2\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z})$$

by [BL03, Lem. 3.1.4]³. On $\ker \phi_L$ we have the alternating Weil pairing

$$e : \ker \phi_L \times \ker \phi_L \longrightarrow \mathbb{C}^\star$$

see [BL03, p. 160], for the special case of an abelian surface see also [BL03, Ex. 6.7.3]. For $[x] = ([x_i]), [y] = ([y_i]) \in \ker \phi_L$ with respect to the isomorphism in (6.15), the pairing e can be calculated as

$$e([x], [y]) = \exp\left(\frac{2\pi i}{d_1}(x_3 y_1 - x_1 y_3)\right) \cdot \exp\left(\frac{2\pi i}{d_2}(x_4 y_2 - x_2 y_4)\right),$$

see [BL03, Ex. 6.7.3].

Choose a subgroup $G \subset \ker \phi_L$ such that $G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}$ and which is isotropic with respect to the pairing above.

Then ϕ_L factorizes as

$$\begin{array}{ccc} S & \xrightarrow{\phi_L} & S^\vee \\ & \searrow p & \nearrow p^\star \\ & S/G & \end{array}$$

where p is the canonical projection which is a $d = d_1 d_2$ to 1 map. As G is isotropic, by [BL03, Prop. 6.7.1] the action of G on S lifts to a free action of G on L , in particular we can define $L_0 := L/G \in \text{Pic}(S/G)$. Since p is of degree d and $p^\star L_0 = L$, we have

$$2d = (L, L) = (p^\star L, p^\star L) = d(L_0, L_0),$$

i.e. $(L_0, L_0) = 2$, therefore L_0 is a principal polarization on S/G . Hence, $H^0(S/G, L_0) = \mathbb{C}\sigma$ for a nontrivial section σ . Define the curve $C_0 := V(\sigma)$. Then C_0 is an element of $|L_0|$ and we claim that C_0 is smooth and irreducible.

Indeed, assume $C_0 = C_1 + C_2$. Since $\rho(S/G) = 1$ we have $C_1 = m_1 L_0$ and $C_2 = m_2 L_0$ with positive integers m_i . Then

$$2 = C_0^2 = (C_1 + C_2)^2 = (m_1 + m_2)^2 (L_0, L_0) = 2(m_1^2 + m_2^2 + 2m_1 m_2) > 2,$$

which is absurd.

If C_0 is not smooth, then let $\nu : \tilde{C}_0 \rightarrow C_0$ be its normalization. For its genus we have $g(\tilde{C}_0) < g_a(C_0) = 2$. If $g(\tilde{C}_0) = 0$, then $\tilde{C}_0 = \mathbb{P}^1$ which is absurd, since $\nu : \mathbb{P}^1 \rightarrow C_0 \hookrightarrow S/G$ would be a non constant regular map which is not possible. If $g(\tilde{C}_0) = 1$ then \tilde{C}_0 would be an elliptic curve which can be seen as an abelian subvariety of S/G after a translation, if necessary. Then \tilde{C}_0 has a complementary abelian subvariety in the sense as above. This would mean $\rho(S/G) \geq 2$ which contradicts $\rho(S/G) = 1$.

³In [BL03] they use the notation $K(L)$ for $\ker \phi_L$.

We conclude that C_0 is irreducible and smooth. In particular, C_0 is of genus 2 and by Lemma 6.9, $(S/G) \cong (S/G)^\vee$ embeds into $\text{Jac}(C_0)$. Both have the same dimension, hence $S/G \cong \text{Jac}(C_0)$.

Set $C := p^{-1}(C_0)$. Then C is an element of $|L|$ as $L = p^*L_0$ and is smooth as p is etale. It has to be connected with a similar argument as above. Assume $C = C_1 \cup C_2$ is a disjoint union. As $\rho(S) = 1$ we have $C_i = m_i L'$ for positive integers m_i where L' is the primitive part of L . Then

$$0 = (C_1, C_2) = (m_1 L', m_2 L') = 2m_1 m_2 \frac{d_2}{d_1} > 0,$$

which is absurd.

Hence, C is a connected smooth curve.

Denote by $\iota : C \hookrightarrow S$ the inclusion, by $q := p|_C = p \circ \iota : C \rightarrow C_0$ the induced d to 1 cover and by Θ_0 the principal polarization on $\text{Jac}(C_0)$. Since $\rho(S) = 1$, also $\rho(S^\vee) = 1$ and $\rho(\text{Jac}(C_0)) = 1$, so we have for the pullback $(p^*)^*L_\delta = k\Theta_0$ for some positive integer k . As p^* is surjective of degree d , taking the self intersection on both sides gives

$$2k^2 = (k\Theta_0, k\Theta_0) = ((p^*)^*L_\delta, (p^*)^*L_\delta) = d(L_\delta, L_\delta) = 2d^2$$

and hence $k = d$ i.e.

$$(6.16) \quad (p^*)^*L_\delta = d\Theta_0.$$

As $q = p \circ \iota$ we have that q^* is the map

$$q^* : \text{Jac}(C_0) \xrightarrow{p^*} S^\vee \xrightarrow{\iota^*} \text{Jac}(C).$$

Since $\rho(\text{Jac}(C_0)) = 1$, we have $(q^*)^*\Theta = a\Theta_0$ for some positive integer a . By [BL03, Lem. 12.3.1] we have that $(q^*)^*\Theta$ is algebraically equivalent to $d\Theta_0$. Therefore $ac_1(\Theta_0) = c_1((q^*)^*\Theta) = dc_1(\Theta_0)$, hence $a = d$ i.e.

$$(6.17) \quad (q^*)^*\Theta = d\Theta_0.$$

Finally write $(\iota^*)^*\Theta = bL_\delta$ for some positive integer b . We have

$$d\Theta_0 \stackrel{(6.17)}{=} (q^*)^*\Theta = (\iota^* \circ p^*)^*\Theta = (p^*)^*(\iota^*)^*\Theta = (p^*)^*(bL_\delta) \stackrel{(6.16)}{=} bd\Theta_0,$$

hence $b = 1$ i.e. $(\iota^*)^*\Theta = L_\delta$.

- We show that the statement holds for every element in $|L|$ but still assume $\rho(S) = 1$ for the Picard number.

Consider the open and connected set $U \subset |L| \cong \mathbb{P}^{d-1}$ such that every element in U corresponds to a smooth curve in S . Let $\mathcal{C} \rightarrow U$ be the associated family of smooth curves. We can take the relative Jacobian

$$\pi_k : X^k := \text{Pic}^k(\mathcal{C}/U) \longrightarrow U$$

of degree $k \in \mathbb{Z}$ of it.

By Lemma 6.12 the genus of \mathcal{C}_t is $g = d + 1$. By considering the image of $(X^d)^{(d)} \rightarrow X^d$, $(x_1, \dots, x_d) \mapsto \sum_i x_i$ which is a divisor in X^d , we obtain a line bundle $\mathcal{M} \in \text{Pic}(X^d)$ such that $\mathcal{M}_t := \mathcal{M}|_{X_t^d}$ is the natural polarization on $X_t^d = \text{Pic}^d(\mathcal{C}_t)$.

Locally we can identify X^d with X^0 , say $X_V^d = \pi_d^{-1}(V) \cong X_V^0 = \pi_0^{-1}(V)$ where $V \subset U \subset |L|$ is chosen connected, by twisting with a line bundle Q^V on $\pi_d^{-1}(V)$ which has degree $-d$ on the fibers \mathcal{C}_t for $t \in V$. Then we obtain on a line bundle $\mathcal{L}^V = \mathcal{M} \otimes Q^V$ on X_V^0 , such that $\mathcal{L}_t^V := \mathcal{L}^V|_{X_t^0}$ is the principal

polarization on $X_t^0 = \text{Jac}(\mathcal{C}_t)$ for $t \in V$. Let $\iota_t : \mathcal{C}_t \hookrightarrow S$ denote the inclusion. Then the self intersection $m_V : V \rightarrow \mathbb{Z}$

$$(6.18) \quad m_V(t) := \left((\iota_t^*)^* \mathcal{L}_t^V, (\iota_t^*)^* \mathcal{L}_t^V \right)$$

of $(\iota_t^*)^* \mathcal{L}_t^V$ is a continuous and integer valued function, therefore must be constant as V is chosen connected.

By the first part we know that there is an element $t_0 \in U$ such that the statement for the curve \mathcal{C}_{t_0} holds. For arbitrary $t_N \in U$, choose a path γ from t_0 to t_N in U . By the discussion above, we can cover γ with finitely many connected open sets V_0, \dots, V_N such that $t_0 \in V_0$ and $t_N \in V_N$ and we have elements $t_i \in V_i \cap V_{i+1}$ for $i = 1, \dots, N-1$. Then the self intersections m_{V_i} and $m_{V_{i+1}}$ must coincide on $V_i \cap V_{i+1}$.

By assumption we have

$$m_{V_0}(t_0) = (L_\delta, L_\delta) = 2d$$

i.e. $m_{V_0} \equiv 2d$. Assume $(\iota_{t_N}^*)^* \mathcal{L}_{t_N} = kL_\delta$ for some positive integer k . Then

$$2d = m_{V_0}(t_0) = m_{V_1}(t_1) = \dots = m_{V_N}(t_N) = k^2 2d$$

i.e. $k = 1$.

- We now consider the general case i.e. let S be with arbitrary Picard number. We have an universal family $p : \mathcal{X} \rightarrow \mathfrak{h}_2$ of (d_1, d_2) -polarized abelian surfaces over Siegel's upper half plane \mathfrak{h}_2 , see [BL03, 8.7]. Let \mathcal{N} denote the line bundle on \mathcal{X} such that $\mathcal{N}_s := \mathcal{N}|_{\mathcal{X}_s}$ is the (d_1, d_2) polarization on \mathcal{X}_s for $s \in \mathfrak{h}_2$.

For each $s \in \mathfrak{h}_2$ let $U_s \subset |\mathcal{N}_s| \cong \mathbb{P}^{d-1}$ be the open set such that all elements in U_s corresponds to smooth curves in \mathcal{X}_s . Let $U \subset \mathbb{P}^{d-1}$ denote the open and connected subset such that for every $(s, t) \in \mathfrak{h}_2 \times U$ the point $t \in \mathbb{P}^{d-1} \cong |\mathcal{N}_s|$ corresponds to an element in U_s . In particular it corresponds to a smooth curve \mathcal{C}_t^s in \mathcal{X}_s . Let $\iota_{s,t} : \mathcal{C}_t^s \hookrightarrow \mathcal{X}_s$ denote the inclusion.

From the second step of the proof we know that for each $(s, t) \in \mathfrak{h}_2 \times \mathbb{P}^{d-1}$ we can find a neighbourhood $V_{s,t} \subset U_s$ of t and a relative principal polarization $\mathcal{L}^{s,t}$ on $\text{Pic}^0(\mathcal{C}^s/U_s)|_{V_{s,t}}$ where $\mathcal{C}^s \rightarrow U_s$ denotes the associated family of smooth curves to U_s .

We can define the map

$$\varphi : \mathfrak{h}_2 \times U \longrightarrow \mathbb{Z}^2, \quad (s, t) \longmapsto \underline{d} \left((\iota_{s,t}^*)^* \mathcal{L}_t^{s,t} \right)$$

for the case that $(s, t) \in \mathfrak{h}_2 \times V_{s,t}$. This is well defined and continuous, therefore must be constant as U is connected. It is well known, see [BL03, 8.11, (1)], that the generic abelian surface has endomorphism ring $\text{End} = \mathbb{Z}$ i.e. has Picard number $\rho = 1$, by Lemma 6.25. Therefore the statement proven in the second step applies for a generic element $(s_0, t_0) \in \mathfrak{h}_2 \times U$ i.e. $\varphi(s_0, t_0) = (d_1, d_2) \equiv \varphi$.

For our original situation this means that the type of $\Theta|_{S^\vee} = (\iota^*)^* \Theta$ is $\underline{d}(\Theta|_{S^\vee}) = (d_1, d_2)$ for arbitrary (d_1, d_2) -polarized (S, L) .

□

An immediate consequence of Lemma 6.14 and Proposition 6.6 is the following.

Proposition 6.19 *Let (S, L) denote a polarized abelian surface of type $\underline{d}(L) = (d_1, d_2)$. Then for every smooth curve $C \in |L|$, we have that the type of the restriction of the principal polarization Θ of $\text{Jac}(C)$ to $K(C)$ is*

$$\underline{d}(\Theta|_{K(C)}) = (1, \dots, 1, d_1, d_2).$$

Proof: By Lemma 6.14 the restriction $\Theta|_{S^\vee}$ is a polarization of type $\underline{d}(\Theta|_{S^\vee}) = (d_1, d_2)$. By Proposition 6.6 the type of $\Theta|_{K(C)}$ is $\underline{d}(\Theta|_{K(C)}) = (1, \dots, 1, d_1, d_2)$. \square

6.20. Fibers of Beauville–Mukai systems. Let $\pi : K_H(v) \rightarrow |D|$ denote a Beauville–Mukai system of generalized Kummer type. Consider a smooth curve $C \in |D|$, then the fiber of the support morphism $M_H(v) \rightarrow \{D\}$ is given by the Jacobian $\text{Jac}^d(C)$ of a certain degree d . The restriction of the Albanese map $(\text{Alb}_v)_{F_0} = \alpha_{F_0} \times \det_{F_0}$ to $\text{Jac}^d(C) \subset M_H(v)$ is in the second component constant 0. Therefore, if we denote by $K^d(C) \subset \text{Jac}^d(C)$ the fiber of $\pi : K_H(v) \rightarrow |D|$, we have an exact sequence

$$(6.21) \quad 0 \longrightarrow K^d(C) \hookrightarrow \text{Jac}^d(C) \xrightarrow{\alpha} S$$

where $\alpha = \text{pr}_S \circ (\text{Alb}_v)_{F_0}$ and the diagram

$$(6.22) \quad \begin{array}{ccccc} K_H(v) & \hookrightarrow & M_H(v) & \xrightarrow{(\text{Alb}_v)_{F_0}} & S \times S^\vee \\ \uparrow & & \uparrow & & \uparrow \\ K^d(C) & \hookrightarrow & \text{Jac}^d(C) & \xrightarrow{\alpha} & S \end{array}$$

Lemma 6.23 *The map $\alpha = \text{Jac}(\iota)$ above is the map induced by the inclusion $\iota : C \hookrightarrow S$ by the universal property of the Jacobian. More precisely α is given by*

$$\mathcal{O}_C(\sum_i n_i x_i) \mapsto \sum_i n_i x_i.$$

In particular, $K^d(C)$ is the kernel of this map.

Proof: This follows from the definition of the map α , see (6.1). If $F \in \text{Jac}^d(C) \subset M_H(v)$, then α takes on $\text{Jac}^d(C)$ the form $\mathcal{O}_C(\sum_i n_i x_i) \mapsto \sum_i n_i x_i$ which is the map induced by ι and the universal property of the Jacobian, see subsection 6.7. The second statement is obvious. \square

By Lemma 6.9 we know that we can see the dual $S^\vee = \text{Pic}^0(S)$ as an abelian subvariety of $\text{Jac}^d(C)$, as the pullback $\iota^* : \text{Pic}^0(S) \hookrightarrow \text{Jac}(C) \cong \text{Jac}^d(C)$ is an embedding. We conclude that we are in the situation of 6.7 and therefore have the following.

Proposition 6.24 *In the situation above, $K^d(C)$ and S^\vee are a pair of complementary abelian subvarieties in the principally polarized abelian variety $\text{Jac}^d(C)$.*

Proof: Follows immediately from Lemma 6.11. \square

Lemma 6.25 [Wiel6, Lem. 5.4] *Let A be an abelian variety. If $\text{End}(A) = \mathbb{Z}$ then its Picard number is $\rho(A) = 1$.*

We now consider Jacobians of curves which are contained in linear systems defined on abelian surfaces.

Theorem 6.1 [CvdG92, 3.B.] *Let S be an abelian surface, $\iota : C \hookrightarrow S$ a smooth curve and let denote by*

$$K(C) := \ker(\text{Jac}(C) \rightarrow S) \subset \text{Jac}(C)$$

the kernel of the map $\text{Jac}(\iota)$ induced by the inclusion and the universal property of the Jacobian, as described in subsection 6.7. Then $\text{End}(K(C)) = \mathbb{Z}$, therefore we have for the Picard number $\rho(K(C)) = 1$.

Proof: We know by Lemma 6.9 that $K(C)$ is connected i.e. a honest abelian subvariety of $\text{Jac}(C)$. The requirement in [CvdG92, 2.II.] that $|C|$ defines a birational map on its image can be dropped, since the authors only use this to conclude that the map $\iota^* : S^\vee \rightarrow \text{Jac}(C)$ has finite kernel. In our setting this is the case by Lemma 6.9. Then by [CvdG92, 3.B.] we have $\text{End}(K(C)) = \mathbb{Z}$, hence $\rho(K(C)) = 1$ for the Picard number by Lemma 6.25. \square

Further we can compute the polarization types of Beauville–Mukai systems of generalized Kummer type.

Theorem 6.2 *The Picard number of the generic smooth fiber of a Beauville–Mukai system $\pi : X \rightarrow |D|$ of generalized Kummer n -type equals one. In particular we have for its polarization type*

$$\underline{d}(\pi) = (1, \dots, 1, d_1, d_2)$$

where $\underline{d}(D) = (d_1, d_2)$ is the type of the polarization defined by D .

Proof: Let us denote $C \in |D|$ a generic smooth curve. The fiber $F = K(C) = K^d(C)$ of π over C is given as the kernel of the map $\text{Jac}(\iota) : \text{Jac}^d(C) \rightarrow S$, see (6.21), where $\iota : C \hookrightarrow S$ is the inclusion. We are therefore precisely in the situation of Theorem 6.1 which states that $\rho(K(C)) = 1$ for the Picard number. Let $\omega \in \mathcal{K}_X$ denote a special Kähler class for the fiber $K(C)$. We are in the case of subsection 6.7 and by Proposition 6.19 the abelian subvariety $K(C)$ admits a polarization L of type $\underline{d}(L) = (1, \dots, 1, d_1, d_2)$. Since $\rho(K(C)) = 1$, we have $L = \omega|_{K(C)}$ as both are primitive. Therefore $\underline{d}(\pi) = \underline{d}(\omega|_F) = \underline{d}(L) = (1, \dots, 1, d_1, d_2)$ by Proposition 3.1. \square

6.26. Beauville–Mukai systems in the moduli. In this section we show that there are Beauville–Mukai systems in each connected component of the moduli of Lagrangian fibrations of $\text{K3}^{[n]}$ and generalized Kummer type. We check this in terms of the monodromy invariant.

The proof of the following Proposition is similar to [Mar14, Ex. 3.1]. However, we give a detailed proof.

Proposition 6.27 *Let d be a positive integer such that d^2 divides $n + 1$ and let b an integer satisfying $\gcd(d, b) = 1$. Then there exists a Beauville–Mukai system $\pi : K_H(v) \rightarrow \mathbb{P}^n$ of generalized Kummer type and a primitive isotropic class $\alpha \in H^2(K_H(v), \mathbb{Z})$ such that the following holds.*

- (i) $\text{Div}(\alpha) = d$,
- (ii) the monodromy invariant $\vartheta(\alpha)$ is represented by $(L_{n,d}, (d, b))$,
- (iii) $c_1(\pi^* \mathcal{O}_{\mathbb{P}^n}(1)) = \alpha$.
- (iv) Its polarization type is given by $\underline{d}(\pi) = (1, \dots, 1, d, \frac{n+1}{d})$.

Proof: Let S be an abelian surface together with primitive ample line bundle L on S with $(L, L) = (2n + 2)/d^2$. Set $\beta := c_1(L)$ and let s be an integer such that $sb \equiv 1 \pmod{d}$. Then $v := (0, d\beta, s)$ is a Mukai vector. In particular v is primitive since β

is primitive and $\gcd(d, s) = 1$. Choose a v -generic ample class H . We have $(v, v) = d^2(\beta, \beta) = 2n + 2$ hence $K_H(v) \subset M_H(v)$ is irreducible holomorphic symplectic of dimension $2n$ and we obtain a Beauville–Mukai system $\pi : M_H(v) \rightarrow |L^d|$ as described in section 6. We have Mukai’s Hodge isometry

$$\Theta : v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z}) \xrightarrow{r} H^2(K_H(v), \mathbb{Z})$$

see (4.7) and (4.8). The map $r : H^2(M_H(v), \mathbb{Z}) \longrightarrow H^2(K_H(v), \mathbb{Z})$ is the restriction. Recall that the definition of Θ needs the choice of a quasi-universal family of sheaves \mathcal{E} on S of similitude $\rho \in \mathbb{N}$.

Set $\alpha := \Theta(0, 0, 1)$ which is clearly isotropic and define $\iota : H^2(K_H(v), \mathbb{Z}) \rightarrow H^\bullet(S, \mathbb{Z})$ to be Θ^{-1} composed with the inclusion $v^\perp \hookrightarrow H^\bullet(S, \mathbb{Z})$. Note that ι is a representative of the monodromy invariant orbit constructed in Theorem 4.1.

(i) An element (r, c, t) belongs to v^\perp iff

$$0 = ((0, d\beta, s), (r, c, t)) = d(\beta, c) - rs \iff rs = d(\beta, c).$$

Hence d divides r since $\gcd(d, s) = 1$. Further we have $((0, 0, 1), (r, c, t)) = r$ for all $(r, c, t) \in v^\perp$ hence $\text{Div}((0, 0, 1)) \geq d$. As the lattice of a two torus is $\mathbb{U}^{\oplus 3}$ i.e. in particular unimodular, we have $\text{Div}_{H^2(S, \mathbb{Z})}(\beta) = 1$ in $H^2(S, \mathbb{Z})$. This implies that $\text{Div}(\beta) = 1$ in v^\perp , hence we can find an element $c \in H^2(S, \mathbb{Z})$ such that $s = (c, \beta)$. Then $(d, c, 0)$ is contained in v^\perp and $((0, 0, 1), (d, c, 0)) = d$, hence

$$\text{Div}(\alpha) = \text{Div}(0, 0, 1) = d.$$

(ii) We have $\iota(\alpha) - bv = (0, 0, 1) - (0, bd\beta, bs) = (0, bd\beta, 1 - bs)$ which is divisible by d since $sb \equiv 1 \pmod{d}$ by assumption. By Lemma 5.12 (v) the monodromy invariant $\vartheta(\alpha)$ is represented by $(L_{n,d}, (d, b))$.

(iii) Let $\omega = [p] \in H^4(S, \mathbb{Z})$ denote Poincaré dual of a point $p \in S$. By our notation we have $\omega = (0, 0, 1) = \omega^\vee \in H^\bullet(S)$. Since S is an abelian surface, one has $\sqrt{\text{Td}(S)} = 1$, hence $\sqrt{\text{Td}(S)}\omega = \omega$. Note that \mathcal{E} is a sheaf of rank zero, hence $\text{ch}(\mathcal{E}) = \rho c_1(\mathcal{E}) + \xi = \rho[D] + \xi$ for some divisor D in $S \times M_H(v)$ and for some terms ξ of higher degree. Further $(\text{pr}_S)^*\omega = [p \times M_H(v)] \in H^4(S \times M_H(v), \mathbb{Z})$ and $[(\text{pr}_{M_H(v)})!(\xi \cdot [p \times M_H(v)])]_2 = 0$ due to degree reasons. Then we have

$$\begin{aligned} \Theta(0, 0, 1) &= r \left((\text{pr}_{M_H(v)})! (D \cdot [p \times M_H(v)]) \right) \\ &= r ([F \in M_H(v) \mid p \in \text{supp}(F)]) \\ &= [F \in K_H(v) \mid p \in \text{supp}(F)] \\ &= \pi^*[C \in |L^d| \mid p \in C] \\ &= \pi^*c_1(\mathcal{O}_{|L^d|}(1)) = c_1(\pi^*\mathcal{O}_{|L^d|}(1)) \end{aligned}$$

since $V := \{C \in |L^d| \mid p \in C\}$ is a hyperplane in a projective space, hence $[V] = c_1(\mathcal{O}_{|L^d|}(1))$.

(iv) This follows directly from Theorem 6.2 since $\underline{d}(L) = (1, \frac{n+1}{d^2})$ by Lemma 6.12 i.e. $\underline{d}(dL) = (d, \frac{n+1}{d})$.

□

6.28. Geometric interpretation of the monodromy invariant. As in the $\text{K3}^{[n]}$ -case we have the following *connected component of the moduli of generalized Kummer fibrations*.

Let Λ denote a lattice of signature $(3, b_2 - 3)$ which is isometric to the second integral cohomology of an irreducible holomorphic symplectic manifold.

Let \mathfrak{M}_Λ denote the corresponding moduli space of isomorphism classes of marked pairs (X, η) i.e. X is an irreducible holomorphic symplectic manifold of the fixed deformation type and $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ is a marking. Choose a connected component $\mathfrak{M}_\Lambda^\circ$ of \mathfrak{M}_Λ and consider the period map

$$\mathcal{P} : \mathfrak{M}_\Lambda^\circ \longrightarrow \Omega_\Lambda, \quad (X, \eta) \longmapsto [\eta(H^{2,0}(X))].$$

Choose the orientation of $\tilde{\mathcal{C}}_\Lambda$ compatible to $\mathfrak{M}_\Lambda^\circ$ in sense of Definition 2.20.

Let $\lambda \in \Lambda$ be a nontrivial isotropic class. After a possible change of the sign of λ (cf. 2.22), we have a distinguished and compatible connected component

$$\Omega_{\lambda^\perp}^+ := \{p \in \Omega_{\lambda^\perp} \mid \lambda \in \partial\mathcal{C}_p\}$$

of the hyperplane section $\Omega_{\lambda^\perp} = \Omega_\Lambda \cap \lambda^\perp$, see 2.22. Then define

$$\mathfrak{M}_{\lambda^\perp}^\circ := \mathcal{P}^{-1}(\Omega_{\lambda^\perp}^+) = \left\{ (X, \eta) \in \mathfrak{M}_\Lambda^\circ \mid \eta^{-1}(\lambda) \text{ is of type } (1, 1) \text{ and in } \partial\mathcal{C}_X \right\}.$$

which is a connected hypersurface of $\mathfrak{M}_\Lambda^\circ$ by [Mar14, Lem. 4.4] and [Mar13, Cor. 5.11]. Consider the nef subspace

$$\mathfrak{U}_{\lambda^\perp}^\circ := \left\{ (X, \eta) \in \mathfrak{M}_{\lambda^\perp}^\circ \mid \eta^{-1}(\lambda) \text{ is nef} \right\}.$$

As in the $K3^{[n]}$ -type we have the following result for the generalized Kummer case, which can be proved exactly in the same way as in the $K3^{[n]}$ case with use of [Mat13, Cor. 1.1].

Theorem 6.3 [Wie16, Thm. 3.7] *Let λ be a primitive and isotropic element in the $K3^{[n]}$ or generalized Kummer lattice. The space $\mathfrak{U}_{\lambda^\perp}^\circ$ in the corresponding connected component $\mathfrak{M}_\Lambda^\circ$ of the moduli of marked pairs has the following properties.*

- (i) *It parametrizes isomorphism classes of marked pairs (X, η) of $\mathfrak{M}_\Lambda^\circ$ with X of $K3^{[n]}$ or generalized Kummer type, respectively, admitting a Lagrangian fibration $f : X \rightarrow \mathbb{P}^n$ such that*

$$\eta(c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1))) = \lambda.$$

- (ii) *It is smooth of dimension 20 for the $K3^{[n]}$ and of dimension 4 for the generalized Kummer case. Further it is open in $\mathfrak{M}_{\lambda^\perp}^\circ$.*
- (iii) *It is connected.*

We refer to this space $\mathfrak{U}_{\lambda^\perp}^\circ$ as a *connected component of the moduli of Lagrangian fibrations*.

We can state the geometric interpretation of the monodromy invariant.

Proposition 6.29 *Let $f_i : X_i \rightarrow \mathbb{P}^n$, $i = 1, 2$, denote two Lagrangian fibrations of generalized Kummer type. Let Λ denote the generalized Kummer lattice and set $L_i := f_i^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then the following statements are equivalent.*

- (i) *The Lagrangian fibrations f_i are deformation equivalent.*
- (ii) *There exist markings $\eta_i : H^2(X_i, \mathbb{Z}) \rightarrow \Lambda$ such that the marked pairs (X_i, η_i) are contained in the same connected component $\mathfrak{U}_{\lambda^\perp}^\circ$ for a primitive isotropic in the generalized Kummer lattice.*
- (iii) [Mar13, Lem. 5.17] *We have $\text{Div}(c_1(L_1)) = \text{Div}(c_1(L_2))$ for the corresponding divisibilities and $\vartheta(c_1(L_1)) = \vartheta(c_1(L_2))$ for the monodromy invariant.*

7. Polarization types of generalized Kummer fibrations

We have now gathered everything to compute the polarization type of Lagrangian fibrations of generalized Kummer type.

Theorem 7.1 *Let $f : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration of generalized Kummer type. Then for the polarization type $\underline{d}(f)$ we have*

$$\underline{d}(f) = \left(1, \dots, 1, d, \frac{n+1}{d}\right)$$

where $d := \text{Div}(c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1)))$ denotes the divisibility of the associated element in the lattice $H^2(X, \mathbb{Z})$.

Proof: Let $f : X \rightarrow \mathbb{P}^n$ denote a Lagrangian fibration of generalized Kummer type and set $L := f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then $\lambda := c_1(L)$ is primitive and isotropic by Lemma 2.7 with respect to the Beauville–Bogomolov quadratic form. Let $d := \text{Div}(\lambda)$ denote the divisibility of λ , note that by Lemma 5.12 d^2 divides $n+1$. Consider the monodromy invariant $\vartheta : I_d(X) \rightarrow \Sigma_{n,d}$ as in Theorem 5.1. By Lemma 5.12 (v) there exists an integer b such that $\vartheta(\lambda)$ is represented by $(L_{n,d}, (d, b))$ and we have $\gcd(d, b) = 1$.

By Theorem 6.27 we have a Beauville–Mukai system $\pi : X' \rightarrow \mathbb{P}^n$ of generalized Kummer type, respectively, together with a primitive isotropic class $\alpha \in H^2(X', \mathbb{Z})$ such that $\text{Div}(\alpha) = d$, $L' := \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ satisfies $c_1(L') = \alpha$ and $\vartheta(\alpha)$ is represented also by $(L_{n,d}, (d, b))$ i.e. $\vartheta(\alpha) = \vartheta(\lambda)$.

Further by Lemma 5.5 we have $(\omega, L) > 0$ and $(\omega', L') > 0$ for Kähler classes ω on X and ω' on X' as L and L' are isotropic and nef, therefore are contained in $\bar{\mathcal{K}}_X \subset \bar{\mathcal{C}}_X$ and $\bar{\mathcal{K}}_{X'} \subset \bar{\mathcal{C}}_{X'}$ respectively. Hence we can apply Lemma 5.4 to see that the pairs (X, L) and (X', L') are deformation equivalent in the sense of Definition 2.4. By Proposition 2.6 the Lagrangian fibrations π and f are deformation equivalent. By Theorem 3.1 and Theorem 6.27 we have

$$\underline{d}(f) = \underline{d}(\pi) = \left(1, \dots, 1, d, \frac{n+1}{d}\right)$$

which concludes the proof. \square

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